Infinitesimal Models of Algebraic Theories and their Applications (extended abstract)

Filip Bár

Centre of Mathematical Sciences, Lund University

Differential geometry, as the name suggests, is the *geometry of differentials*; that is of *infinitesimals*. Indeed, the infinitesimal calculus as developed by Leibniz was the major tool for developing calculus and differential geometry until the beginning of 20th century. However, its major drawback was that infinitesimals eluded a formal definition until the advent of Non-Standard Analysis and Synthetic Differential Geometry (SDG); the latter only made possible due to the versatility of topos-theoretic constructions guaranteeing the existence of models [5, 7, 9, 6].

In the framework of SDG it is possible to make Leibniz's infinitesimal calculus precise by extending the classical notion of a model of an algebraic theory to that of an *infinitesimal model* as has been done in [4] and [1]. As an algebra of infinitesimal pieces of space infinitesimal algebra is able to clarify and simplify many of the standard differential geometric constructions by drawing on classic algebraic structures like affine space, vector space, group and Lie algebra (see [2], [3] and [10]). The category of infinitesimal models of an algebraic theory retains most of the good categorical properties of categories of models of algebraic theories, but with much better gluing properties. This makes infinitesimal models an interesting natural construction with the ability to interpolate between algebra and geometry. It would be interesting to see applications outside the realm of SDG.

Background In [8] Kock has shown that a (formal) manifold in Synthetic Differential Geometry admits affine combinations of points that are pairwise mutual infinitesimal neighbours. In this and subsequent work [10], [11] he has made an extensive use of this geometric algebra of infinitesimally affine combinations linking it with well-known concepts and constructions from Differential Geometry. Building on Kock's work the author has been trying to understand in which sense formal manifolds are models of affine spaces and whether this can be extended to other algebraic theories like groups and vector spaces. This has led him to formulate the notion of an *infinitesimal model of an algebraic theory* as a space equipped with an *infinitesimal structure* (i-structure) that serves as the domain of the operations of the theory in [4] and [1].

Infinitesimal models of algebraic theories Similar to the structure of a topology an i-structure defines the collections of infinitesimally neighbouring points. However, in contrast to open sets the collections are finite tuples of points. This allows us to use an infinitesimal structure as a domain of operations of a (finitary) algebraic theory $\mathbb{T}$.

There are two equivalent ways to define an infinitesimal model of $\mathbb{T}$:

(1) *The technically convenient:* Identify $\mathbb{T}$ with its abstract clone $O_{\mathbb{T}}$ of operations and define an infinitesimal model as the action of $O_{\mathbb{T}}$ on an i-structure.

(2) *The pragmatical:* Starting with a presentation of $\mathbb{T}$ construct its *infinitesimalisation* $\mathrm{I}[\mathbb{T}]$ and define an infinitesimal model of $\mathbb{T}$ as the model of the cartesian theory $\mathrm{I}[\mathbb{T}]$.

The key axiom for infinitesimal models is the *neighbourhood axiom*, which guarantees that when starting with an infinitesimal neighbourhood of points all the $\mathbb{T}$-operations yield neighbourhoods and are thus composable.

As a category of models of a cartesian theory $\mathrm{I}[\mathbb{T}]$ the category of infinitesimal $\mathbb{T}$-models is locally (finitely) presentable. Moreover, the forgetful functor from the category of infinitesimal $\mathbb{T}$ models to the Grothendieck base topos (which we, for simplicty, think of as **Set**) has a remarkable lifting property: it lifts wide pushouts of i-structure reflecting morphisms (which is a very mild condition). This leads to infinitesimal models having some remarkable gluing properties: For example, in stark contrast to $\mathbb{T}$-models, coproducts become essentially unions of the underlying spaces, while coequalizers are, in general, constructed from quotients of congruences like in the category of $\mathbb{T}$-models. Which other colimits are lifted depends on how 'big' the infinitesimal structures of the models are in relation to each other, and we provide some sufficient conditions for that.

Applications Current applications of infinitesimal algebra are exclusive to SDG. In [2] the author has shown that a symmetric affine connection is equivalent to an infinitesimal model of affine combinations on the second-order i-structure of a manifold. In particular, manifolds admit *different* infinitesimal models of an affine space on the same i-structure, and we require the use of *different* i-structures beyond the canonical i-structure induced by the first-order neighbourhood of the diagonal used in [10]. This justifies to consider infinitesimal models as structures rather than properties of a manifold.

In [3] this equivalence has been been extended to non-symmetric affine connections, where the author has shown that affine connections correspond to infinitesimal group structures on the second-order monad at every point of the manifold. These groups are abelian (and coincide with the second-order infinitesimal linear structure), if and only if the affine connection is symmetric. Moreover, the two canonical affine connections on a Lie group are all induced by the group structure and the Lie bracket is really just the commutator familiar from matrix groups.

References

[1] F. Bár. *Infinitesimal Models of Algebraic Theories*. PhD thesis, Cambridge University, Department of Pure Mathematics and Mathematical Statistics, 2017. URL `https://doi.org/10.17863/CAM.13058`.

[2] F. Bár. Affine connections and second-order affine structures. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, LXIII(1):35–58, 2022. URL `http://cahierstgdc.com/wp-content/uploads/2022/01/BAR-LXIII-1-2.pdf`.

[3] F. Bár. Second-order infinitesimal groups and affine connections. *ArXiv e-prints*, arXiv:2305.04601, 2023. URL `https://doi.org/10.48550/arXiv.2305.04601`.

[4] F. Bár. On the gluing of infinitesimal models of algebraic theories. *ArXiv e-prints*, arXiv:2305.01357, 2023. URL `https://doi.org/10.48550/arXiv.2305.01357`.

[5] E. J. Dubuc. Sur les modèles de la géométrie différentielle synthétique. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 20(3):231–279, 1979. URL http://www.numdam.org/article/CTGDC_1979__20_3_231_0.pdf.

[6] A. Kock. Affine combinations in affine schemes. *ArXiv e-prints*, arXiv:1508.04322, 18th August, 2015. URL https://doi.org/10.48550/arXiv.1508.04322.

[7] A. Kock. Properties of well-adapted models for synthetic differential geometry. *Journal of Pure and Applied Algebra*, 20(1):55–70, 1981.

[8] A. Kock. Geometric construction of the levi-civita parallelism. *Theory and Applications of Categories*, 4(9):1–16, 1998. URL http://www.tac.mta.ca/tac/volumes/1998/n9/n9.pdf.

[9] A. Kock. *Synthetic Differential Geometry*. Number 333 in London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2nd edition, 2006.

[10] A. Kock. *Synthetic Geometry of Manifolds*. Number 180 in Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2009.

[11] A. Kock. Integration of 1-forms and connections. *ArXiv e-prints*, arXiv:1902.11003, 2019. URL https://doi.org/10.48550/arXiv.1902.11003.