Initial algebras for topologically enriched multi-sorted algebraic theories

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Classically, *multi-sorted algebraic theories* and their *initial* (or *free*) *algebras* have been fundamental in mathematics and computer science. Within the latter field, they have been prominently employed in studying algebraic specification and abstract algebraic datatypes [12], computational effects [13], and algebraic databases [14, 15]. Given a set $\mathcal{S}$ of *sorts*, a *(classical)* $\mathcal{S}$*-sorted signature* is a set Σ of *operation* (or *function*) *symbols* equipped with an assignment to each $\sigma \in \Sigma$ of a finite tuple $(S_1, \ldots, S_n)$ of *input sorts* and an *output sort* $S \in \mathcal{S}$, in which case we write $\sigma : S_1 \times \ldots \times S_n \to S$. A Σ*-algebra* $\mathbb{A}$ is an $\mathcal{S}$-indexed family of *carrier sets* $\mathcal{A} = (A_S)_{S \in \mathcal{S}}$ equipped with, for each $\sigma : S_1 \times \ldots \times S_n \to S$ in Σ, a function

$$\sigma^{\mathbb{A}} : A_{S_1} \times \ldots \times A_{S_n} \to A_S.$$

A *(classical)* $\mathcal{S}$*-sorted algebraic theory* is a pair $\mathcal{T} = (\Sigma, \mathcal{E})$ consisting of a classical $\mathcal{S}$-sorted signature Σ and a set $\mathcal{E}$ of *equations* between Σ*-terms*, and a $\mathcal{T}$*-algebra* is a Σ-algebra that satisfies these equations. Writing $\mathcal{T}\text{-}\mathsf{Alg}$ for the category of $\mathcal{T}$-algebras, the forgetful functor $U^{\mathcal{T}} : \mathcal{T}\text{-}\mathsf{Alg} \to \mathsf{Set}^{\mathcal{S}}$ that sends a $\mathcal{T}$-algebra to its underlying $\mathcal{S}$-indexed family of carrier sets is well known to have a left adjoint $F^{\mathcal{T}} : \mathsf{Set}^{\mathcal{S}} \to \mathcal{T}\text{-}\mathsf{Alg}$, which can be explicitly described as follows. Given an $\mathcal{S}$-indexed family of sets $\mathcal{A} = (A_S)_{S \in \mathcal{S}}$ and a sort $S \in \mathcal{S}$, the carrier set of the free $\mathcal{T}$-algebra $F^{\mathcal{T}}\mathcal{A}$ at the sort S is the set of *terms of sort* S built from the operation symbols of Σ and the elements of $\mathcal{A}$ (regarded as new *constant symbols*), quotiented by a certain *congruence relation* determined by the equations of $\mathcal{T}$. The *(absolutely) free* or *initial* $\mathcal{T}$*-algebra* can then be obtained as the free $\mathcal{T}$-algebra on the $\mathcal{S}$-indexed family constant at the empty set; the initial $\mathcal{T}$-algebra is often viewed as the "intended model" of the multi-sorted algebraic theory $\mathcal{T}$.

The purpose of the present work is to generalize the foregoing from the classical (Set-*enriched*) context to the context of enrichment in a symmetric monoidal category $\mathscr{V} = (\mathscr{V}, \otimes, I)$ that is equipped with a *topological (forgetful) functor* $|-| : \mathscr{V} \to \mathsf{Set}$ (in the sense of [1, §21]) that is strict monoidal (with respect to the cartesian monoidal structure on Set). Prominent examples of such $\mathscr{V}$ include:

- Various categories of topological spaces (which were previously considered for the present purposes in [3]), as well as the category of measurable spaces.

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- The categories of models of *relational Horn theories without equality*, including the category of preordered sets and monotone functions, and the category of (extended) pseudometric spaces and non-increasing functions.

- The categories of *quasispaces* (a.k.a. *concrete sheaves*) on *concrete sites*, which have recently attracted significant interest in the semantics of programming languages [6, 11], and include the categories of diffeological spaces, quasi-Borel spaces [6], bornological sets, (abstract) simplicial complexes, and convergence spaces [2, 4].

Given such a category $\mathscr{V}$ and a set $\mathcal{S}$ of sorts, we define a notion of *$\mathscr{V}$-enriched $\mathcal{S}$-sorted signature* Σ, which extends the notion of classical $\mathcal{S}$-sorted signature by requiring that each operation symbol $\sigma \in \Sigma$ be equipped also with a *parameter object* P of $\mathscr{V}$. A *Σ-algebra* $\mathbb{A}$ is then an $\mathcal{S}$-indexed family $\mathcal{A} = (A_S)_{S \in \mathcal{S}}$ of *carrier objects of* $\mathscr{V}$ equipped with, for each $\sigma : S_1 \times \ldots \times S_n \to S$ in Σ with parameter object P, a *$\mathscr{V}$-morphism*

$$\sigma^{\mathbb{A}} : P \otimes (A_{S_1} \times \ldots \times A_{S_n}) \to A_S.$$

A *$\mathscr{V}$-enriched $\mathcal{S}$-sorted algebraic theory* $\mathcal{T}$ is a pair $\mathcal{T} = (\Sigma, \mathcal{E})$ consisting of a $\mathscr{V}$-enriched $\mathcal{S}$-sorted signature Σ and a set $\mathcal{E}$ of Σ-*equations* (generalizing the classical equations), while a *$\mathcal{T}$-algebra* is a Σ-algebra that satisfies these equations. We write $\mathcal{T}\text{-}\mathsf{Alg}$ for the category of $\mathcal{T}$-algebras, which is equipped with a forgetful functor $U^{\mathcal{T}} : \mathcal{T}\text{-}\mathsf{Alg} \to \mathscr{V}^{\mathcal{S}}$ that sends a $\mathcal{T}$-algebra to its underlying $\mathcal{S}$-indexed family of carrier objects of $\mathscr{V}$.

We show that every $\mathscr{V}$-enriched $\mathcal{S}$-sorted algebraic theory $\mathcal{T} = (\Sigma, \mathcal{E})$ has an underlying classical $\mathcal{S}$-sorted algebraic theory $|\mathcal{T}| = (|\Sigma|, |\mathcal{E}|)$. The forgetful functor $U^{\mathcal{T}}$ has a left adjoint $F^{\mathcal{T}} : \mathscr{V}^{\mathcal{S}} \to \mathcal{T}\text{-}\mathsf{Alg}$, and we show that the resulting adjunction $F^{\mathcal{T}} \dashv U^{\mathcal{T}} : \mathcal{T}\text{-}\mathsf{Alg} \to \mathscr{V}^{\mathcal{S}}$ is a *(strict) lifting* of the adjunction $F^{|\mathcal{T}|} \dashv U^{|\mathcal{T}|} : |\mathcal{T}|\text{-}\mathsf{Alg} \to \mathsf{Set}^{\mathcal{S}}$. In particular, given an $\mathcal{S}$-indexed family $\mathcal{A} = (A_S)_{S \in \mathcal{S}}$ of objects of $\mathscr{V}$, the free $\mathcal{T}$-algebra on $\mathcal{A}$ can be realized as the free $|\mathcal{T}|$-algebra on the $\mathcal{S}$-indexed family of sets $|\mathcal{A}| = (|A_S|)_{S \in \mathcal{S}}$, equipped with an appropriate $\mathscr{V}$-structure. We use this result to establish concrete descriptions of free $\mathcal{T}$-algebras, which have an even more explicit and (*countably*) *inductive* character when $\mathscr{V}$ is cartesian closed. The assumption that $\mathscr{V}$ is a topological category over Set crucially enables all of these central results.

We provide several examples of $\mathscr{V}$-enriched multi-sorted algebraic theories, whose initial algebras can now be explicitly described and computed using the foregoing results. When $\mathscr{V}$ is symmetric monoidal *closed*, we also discuss the close connection between $\mathscr{V}$-enriched multi-sorted algebraic theories and (presentations of) certain $\mathscr{V}$-enriched monads studied in [7, 8, 9], as well as with the *relational (single-sorted) algebraic theories* of [5] and the *quantitative (single-sorted) algebraic theories* of [10], which are special (single-sorted) instances of the theories considered here. We intend to pursue applications of the present work to the development of (topologically) enriched treatments of abstract algebraic datatypes, computational effects, and algebraic databases.

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