Completeness for arbitrary finite dimensions of ZXW-calculus, a unifying calculus

Boldizsár Poór[1] Quanlong Wang[1] Razin A. Shaikh[1,2]
Lia Yeh[1,2] Richie Yeung[1,2] Bob Coecke[1]

[1]Quantinuum, 17 Beaumont Street, Oxford, OX1 2NA, United Kingdom
[2]University of Oxford, Oxford, United Kingdom

Abstract

The full versions of the paper is available at: `https://arxiv.org/abs/2302.12135`

The ZX-calculus is a universal graphical language for qubit quantum computation, meaning that every linear map between qubits can be expressed in the ZX-calculus. Furthermore, it is a complete graphical rewrite system: any equation involving linear maps that is derivable in the Hilbert space formalism for quantum theory can also be derived in the calculus by rewriting. It has widespread usage within quantum industry and academia for a variety of tasks such as quantum circuit optimisation, error-correction, and education.

The ZW-calculus is an alternative universal graphical language that is also complete for qubit quantum computing. In fact, its completeness was used to prove that the ZX-calculus is universally complete. This calculus has advanced how quantum circuits are compiled into photonic hardware architectures in the industry.

These two calculi have different applications because the ZW-calculus naturally expresses the summation of diagrams, while the ZX-calculus is more natural for gate-based quantum computing. In the ZXW-calculus, we unify the ZX- and ZW-calculus in a single framework. This unified calculus allows us to utilise techniques native to both of these calculi and offers new insights into previously unexplored areas. Using the ZXW-calculus, graphical-differentiation [3], -integration [3], and -exponentiation [1] were made possible, thus enabling the development of novel techniques in the domains of quantum machine learning and quantum chemistry.

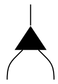
Now several years since the proofs of qubit completeness, the follow-up open question has been the challenge to generalise this to any other quantum information settings. The natural progression would be to consider *qudits* — the generalisation of qubits (which have local dimension 2) to quantum systems with local dimension d.

Here, we generalise the ZXW-calculus to arbitrary finite dimensions, that is, to qudits. Moreover, we prove that this graphical rewrite system is complete for any finite dimension. This is the first completeness result for any universal graphical language beyond qubits. This result shows that it is possible to axiomatise a finite set of rules which generate all equalities of linear maps over any qudit dimensions. It establishes that this is a result not unique to qubits, but holds for a countably infinite family of quantum graphical calculi.

A number of challenges need to be overcome to prove qudit completeness. Knowing complete rule sets for qubits gives little indication as to their extension to higher dimensions. For each generator and rule, there can be numerous ways to formulate different generalisations which coincide for qubits. For instance, squaring any computational state, taking the multiplicative inverse, and multiplying by any odd number (including -1) all evaluate to the identity map on the integers modulo 2. Yet, these are all examples of operations which are almost always non-degenerate with the identity map in any other dimension.

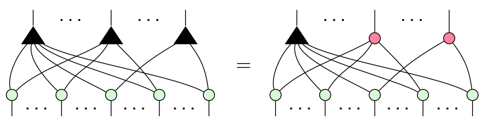
A quick peek at the ZXW-calculus

The Z and X spiders are defined canonically for qudits. The W node has several qudit generalisations, but we chose the following as it has nice interactions with the Z and X spiders:

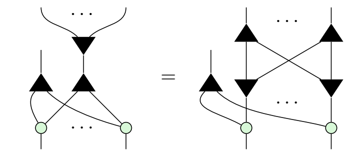


$$\xrightarrow{[\![\cdot]\!]} \quad |00\rangle\langle 0| + \sum_{i=1}^{d-1}(|0i\rangle + |i0\rangle)\langle i|.$$

For example, the *trialbegra* rule presents a beautiful interaction between each of the three, the Z, X and W algebras:

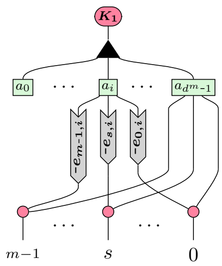


Another interesting rule is the (WW) rule that shows a bialgebra-like interaction between our W node, without the need to introduce the fermionic-swaps:



Approach to the problem

To attain completeness, we adopted an approach similar to the completeness proof of qubit ZW-calculus. We begin by defining a unique normal form that can be used to represent arbitrary complex vectors. Given an arbitrary complex vector $\overrightarrow{a} = (a_0, a_1, \cdots, a_{d^m-1})^T$ of dimension d^m, the normal form [2] is given by



$$\xrightarrow{[\![\cdot]\!]} \quad \sum_{j=0}^{d^m-1} a_j\,|e_{m-1,j}\ \cdots\ e_{s,j}\ \cdots\ e_{0,j}\rangle \ ,$$

where $0 \leq e_{k,j} \leq d-1$.

We show that any diagram can be transformed into an equivalent normal form diagram. To prove this, we first show that any generator rewrites to a diagram in normal form. Next, we prove that the tensor product of normal forms and a partial trace of a normal form can again be transformed into a single unique normal form. Then, given two qudit ZXW diagrams D_1 and D_2 with equal interpretations, we show that $D_1 = D_2$ as follows:

1. Rewrite both D_1 and D_2 into their unique normal form D.
2. Concatenate the rewrite steps from D_1 to D and the rewrite steps from D to D_2.

The above sequence of rewrites demonstrate that $D_1 = D_2$, which proves that the rules of ZXW-calculus are complete for universal qudit computation.

References

[1] Razin A. Shaikh, Quanlong Wang & Richie Yeung (2022): *How to Sum and Exponentiate Hamiltonians in ZXW Calculus.* arXiv:2212.04462.

[2] Quanlong Wang (2022): *Qufinite ZX-calculus: A Unified Framework of Qudit ZX-calculi.* arXiv:2104.06429.

[3] Quanlong Wang, Richie Yeung & Mark Koch (2022): *Differentiating and Integrating ZX Diagrams with Applications to Quantum Machine Learning.* arXiv:2201.13250.