Constructing triple categories of cybernetic processes

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We illustrate a generalized version of the **Para** construction which allows to systematically construct triple categories of cybernetic processes, as well as further extensions thereof to cyebrnetic systems. While **Para** works for actions in categories, our generalization works for any suitably complete 2-category and for more general notions of action (what we call 'oplax dependent actegories'). To exemplify the construction, we show how applying our generalized **Para** to the self-action of a monoidal double category of lenses and charts produces a triple category of parametric lenses, lenses and charts which improves on Spivak and Shapiro's $\mathbb{O}\mathbf{rg}$.

Motivation. In categorical cybernetics, constructing bicategories of parametric maps is a fundamental step in building models of cybernetic systems—simply because a 'cybernetic process' is exactly that: a plant parameterized by a control. The recipe to construct such bicategories (outlined below) has been first described in [2], though this is itself a modified form of the one appearing in [3].

Bicategories of cybernetic processes capture the algebra of such processes, explaining how they can be composed with each other. However, missing from such structures is a way to compare processes among themselves, i.e. a notion of map between cybernetic processes. The work of the second author on *categorical systems theory* [6, 7], which deals with open dynamical systems using a double-categorical framework, shows how powerful this idea is. Indeed, the philosophy of category theory is to study, access and define objects (in our case, cyebrnetic processes) through their morphisms.

We thus set to generalize CST to talk about cybernetic processes and systems, which quickly led to realizing bicategories of cybernetic processes should be extended to triple categories, in the same way categories of processes are extended to double categories in categorical systems theory. Indeed, one can construct 'by hand' triple categories of the most elementary kind of cybernetic processes, parametric lenses.

However we'd like to have a general recipe for constructing such triple categories (and, in the future, triply indexed categories) whenever we have suitable data for it. Therefore, we have been looking at **Para** to see whether it could be fruitfully generalized to this setting.

The generalized Para construction. Recall $\mathbf{Para} : \mathbb{A}\mathbf{ct}(\mathbb{C}\mathbf{at}) \to \mathbb{B}\mathbf{icat}$ is a 2-functorial construction that takes in an actegory $(\mathbf{M}, \mathbf{C}, \odot)$ and returns a bicategory whose 1-cells are $\mathbf{M}$-parametric $\mathbf{C}$-morphisms:

$$(m, f : m \odot x \to y) \in \mathbf{Para}(\odot)(x, y). \tag{1}$$

Our generalization starts from the observation **Para** can be more symmetrically described as a construction $\mathbb{A}\mathbf{ct}(\mathbb{C}\mathbf{at}) \to \mathbb{C}\mathbf{at}(\mathbb{C}\mathbf{at})$. The resulting double category has non-parametric morphisms as tight cells, parametric ones as loose cells, and reparameterizations in the squares:

$$\begin{array}{ccc} x & \xrightarrow{h} & y \\ {\scriptstyle (m,f)}\big\downarrow & \xRightarrow{\alpha} & \big\downarrow{\scriptstyle (n,g)} \\ z & \xrightarrow[k]{} & t \end{array} \quad \equiv \quad \begin{array}{ccc} m \odot x & \xrightarrow{\alpha \odot h} & n \odot y \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ z & \xrightarrow[k]{} & t \end{array} \quad . \tag{2}$$

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This double category turns out to be swiftly definable as a composite pseudomonad in $\mathbf{Span}(\mathbb{C}\mathbf{at})$:

$$\begin{array}{ccccccc} & & & \mathbf{Para}(\odot)_1 & & & \\ & & \swarrow & \lrcorner & \searrow & & \\ & \mathbf{M}\times\mathbf{C} & & & & \mathbf{C}^{\to} & \\ \overset{\pi_{\mathbf{C}}}{\swarrow} & & \overset{\odot}{\searrow} & & \overset{\mathrm{dom}}{\swarrow} & & \overset{\mathrm{cod}}{\searrow} \\ \mathbf{C} & & & \mathbf{C} & & & \mathbf{C} \end{array} \tag{3}$$

Clearly such composite exists when the two pseudomonads we started with distribute over each other, and indeed one realizes this is the case by virtue of the fact $\pi_{\mathbf{C}}$ is a cartesian (Grothendieck) fibration. Therefore one is naturally led to generalize **Para** to take in as data pseudomonads of a more general kind than the ones deriving from actegories.

First of all, we can work in any suitably complete 2-category $\mathbb{K}$ (so far we worked in $\mathbb{K} = \mathbb{C}\mathbf{at}$). Secondly, we define an **oplax dependent actegory** to be a pseudomonad in $\mathbf{f}\mathbb{S}\mathbf{pan}^{\Rightarrow}(\mathbb{K})$, where the latter is the tricategory of spans whose left leg is a fibration in $\mathbb{K}$. The superscript indicates 2-cells in this tricategory only commute up to a non-invertible 2-cell on the right, as well as being compatible with the fibrations on the left:

$$\begin{array}{ccccc} & \overset{\mathrm{fib}}{\swarrow} & X & \searrow & \\ C & & \mathrm{cart.}\big\downarrow \quad \Rightarrow & & C \\ & \underset{\mathrm{fib}}{\nwarrow} & Y & \nearrow & \end{array} \tag{4}$$

The **Para** construction thus composes an oplax dependent actegory with the arrow span $C \overset{\mathrm{dom}}{\leftarrow} C^{\to} \overset{\mathrm{cod}}{\to} C$, which lives in $\mathbf{f}\mathbb{S}\mathbf{pan}^{\Rightarrow}(\mathbb{K})$ as well, and equips the result with a canonical pseudomonad structure.[1] We thus get a functor $\mathbf{Para} : \{\text{oplax dep. actegories}\} \to \mathbb{C}\mathbf{at}(\mathbb{K})$.

Notable examples of oplax dependent actegories in $\mathbb{C}\mathbf{at}$ are, besides actegories, comprehension categories with unit [4], and graded comonads. Instantiating our generalized **Para** for these we recover, respectively, the usual **Para** construction, categories of spans, and coKleisli categories.

Cybernetic processes. As an application, we instantiate the generalized **Para** construction for the self-action of a monoidal double category, i.e. for $\mathbb{K} = \mathbb{D}\mathbf{blCat}$. The resulting object is a category in double categories, hence a triple category.

In particular, from the chief example in [6, 7], that of the monoidal double category $\mathbb{A}\mathbf{rena}$, we obtain a triple category $\mathbf{Para}(\mathbb{A}\mathbf{rena})$ of lenses, charts and parametric lenses. Its cubes can be used to define behaviour for certain cybernetic systems (for instance, Nash equilibria of games can be expressed in this way, as already observed in [1]).

This triple category is related to the double category $\mathbb{O}\mathbf{rg}$ proposed by Spivak and Shapiro in [8] to model controllable dynamical systems. In fact, in our triple category, the 'process-like' and 'simulation-like' cells are better separated and still allow to do what the authors do in $\mathbb{O}\mathbf{rg}$.

References

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