Operads arising from base-valued enriched functors

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Multicategories, also called *operads*, were introduced by Lambek in his categorical studies of logic and linguistics [1, 2]. Multicategories are widely used in applied category theory, as they provide a very general formalism for compositional structures; for example, see [3, 4] and the references therein.

In this talk, we provide a general result that gives rise to a great many examples of symmetric multicategories, both new and old. Given any symmetric multicategory $\mathscr{V}$ and any $\mathscr{V}$-enriched category $\mathscr{C}$ equipped with a suitably $\mathscr{V}$-enriched functor $G : \mathscr{C} \to \mathscr{V}$, we define a notion of G-*multimorphism* that generalizes the notions of multilinear map and $\mathbb{T}$-multimorphism for a commutative monad $\mathbb{T}$ [5], while in our setting G does not even need to be faithful, let alone monadic. A G-multimorphism $f : A_1, A_2, ..., A_n \to B$ for objects $A_1, ..., A_n, B$ of $\mathscr{C}$ is a morphism $|f| : GA_1, ..., GA_n \to GB$ in $\mathscr{V}$ equipped with a family of morphisms $f^{(j)} : GA_1, ..., GA_{j-1}, GA_{j+1}, ..., GA_n \to \mathscr{C}(A_j, B)$ ($j \in \{1, ..., n\}$) in $\mathscr{V}$ satisfying a certain axiom. In the case where the list $A_1, ..., A_n$ is nonempty (i.e. $n > 0$), the morphism $|f|$ is uniquely determined by the $f^{(j)}$.

We show that there is a symmetric multicategory $\mathscr{C}_G$ whose morphisms are the G-multimorphisms. The underlying ordinary category of $\mathscr{C}_G$ is the same as that of $\mathscr{C}$, because a G-multimorphism $f : A_1 \to B$ of arity $n = 1$ is given by a morphism $f^{(1)} : \ \to \mathscr{C}(A_1, B)$ of arity 0 in $\mathscr{V}$. As a corollary, if $\mathscr{C}$ is any locally small category and $G : \mathscr{C} \to \mathrm{Set}$ is any Set-valued functor, then $\mathscr{C}$ underlies a symmetric multicategory $\mathscr{C}_G$ whose morphisms are the G-multimorphisms. In this case where $\mathscr{V} = \mathrm{Set}$, examples of G-multimorphisms for suitable choices of G include (1) multilinear maps, when $\mathscr{C}$ is the category of vector spaces over a field, (2) maps that are continuous in each variable separately, when $\mathscr{C}$ is the category of topological spaces, (3) maps that are 'group homomorphisms in each variable separately', when $\mathscr{C}$ is the category of groups, (4) *sesquifunctors*, when $\mathscr{C}$ is the category of small categories, and (5) graph morphisms defined on the *box product* of several graphs, when $\mathscr{C}$ is any one of several categories of graphs. These examples may all be regarded as special cases of the following further corollary: If C is any object of a locally small category $\mathscr{C}$, then the functor $\mathscr{C}(C, -) : \mathscr{C} \to \mathrm{Set}$ equips $\mathscr{C}$ with the structure of a symmetric multicategory. Examples when $\mathscr{V}$ is a closed category other than Set abound also and include not only versions of certain of the above examples internal to suitable bases $\mathscr{V}$ but also certain unfamiliar but nontrivial examples that arise when $\mathscr{V}$ is the Lawvere half-line, by taking $\mathscr{C}$ to be a normed vector space regarded as a metric space and so as a $\mathscr{V}$-category. But our general result can be applied to any symmetric multicategory $\mathscr{V}$, which need not be closed nor monoidal.

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References

[1] J. Lambek, *Deductive systems and categories. II. Standard constructions and closed categories*, Lecture Notes in Mathematics 86 (1969), 76–122, Springer.

[2] J. Lambek, *On a connection between algebra, logic and linguistics*, Diagrammes 22 (1989), 59–75.

[3] B. Fong, D. I. Spivak, *An invitation to applied category theory. Seven sketches in compositionality.* Cambridge University Press, Cambridge, 2019. xii+338 pp.

[4] J. D. Foley, S. Breiner, E. Subrahmanian and J. M. Dusel, *Operads for complex system design specification, analysis and synthesis*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 477 (2021), Issue 2250.

[5] A. Kock. *Bilinearity and cartesian closed monads*. Mathematica Scandinavica 29 (1971), 161–174.