Optics for Premonoidal Categories

James Hefford

University of Oxford, UK

`james.hefford@cs.ox.ac.uk`

Mario Román

Tallinn University of Technology, Estonia

`mroman@ttu.ee`

We further the theory of optics or "circuits-with-holes" to encompass premonoidal categories: monoidal categories without the interchange law. Every premonoidal category gives rise to an effectful category (i.e. a generalised Freyd-category) given by the embedding of the monoidal subcategory of central morphisms. We introduce "pro-effectful" categories and show that optics for premonoidal categories exhibit this structure.

Pro-effectful categories are the non-representable versions of effectful categories, akin to the generalisation of monoidal to promonoidal categories. We extend a classical result of Day to this setting, showing an equivalence between pro-effectful structures on a category and effectful structures on its free tight cocompletion. We also demonstrate that pro-effectful categories are equivalent to prostrong promonads.

1 Introduction

Monoidal categories play a central role in many categorical models, from programming semantics [4], to quantum theory [2, 16], to electrical circuits [9], for they provide the necessary mathematical structure to describe the interaction of systems over time (by composition) and space (by tensor product). Monoidal categories have a graphical calculus known as string diagrams [30] which provides a formalisation of circuit diagrams. There has been much interest in studying the categorical structure of *circuits-with-holes* [13, 32, 49] over a given monoidal category $\mathscr{C}$; that is, incomplete diagrams in $\mathscr{C}$.

The categorical methods required to describe these circuits-with-holes have their roots in the study of strong profunctors, or *Tambara modules* [47, 7]. These modules are the algebras for a certain promonad [35] with the resulting category of free algebras known as the category of *optics* by the functional programming community, where it is used to model various bidirectional data accessors [15, 40, 41]. The category $\mathsf{Optic}(\mathscr{C})$ has objects given by pairs (a, a') of objects of $\mathscr{C}$ and homs $\mathsf{Optic}(\mathscr{C})((a, a'), (b, b'))$ given by the quotiented sets of the form in Figure 1 [35, 40, 15]. The idea is to produce a category of holes in circuits from $\mathscr{C}$, where two circuits are equivalent if they can be rewritten into each

Figure 1: Optics equivalence relation.

other by sliding boxes. This equivalence relation is handled by the coend $\int^{xy} \mathscr{C}(a, x \otimes b \otimes y) \times \mathscr{C}(x \otimes b' \otimes y, a')$. Outside of functional programming, this category has been suggested as a way to model holes in general monoidal categories [42]. In particular, incomplete diagrams have applications in quantum theory where they are known as *combs* [13] and capture a certain subset of the more general quantum supermaps [14]: optics have been suggested to formalise these structures [25].

What is still missing is a full description of optics for *pre*monoidal categories. Informally, premonoidal categories are like monoidal categories but dropping the interchange law so that in general
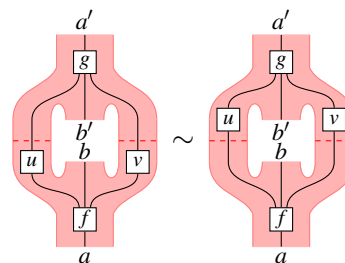
$(1 \otimes f)(g \otimes 1) \neq (g \otimes 1)(1 \otimes f)$. Such categories are useful in the modelling of computational side-effects and it was with precisely this motivation that Power and Robinson introduced them [38]. This manuscript is indebted to the research into premonoidal categories [29, 33, 36, 39, 45].

Since not all morphisms in a premonoidal category interchange, there is now an additional subtlety to formalising optics. The coends that are usually used to quotient to allow for the sliding of morphisms can now only be taken over the *centre $Z\mathscr{C}$* of the premonoidal category $\mathscr{C}$ - the wide monoidal subcategory comprised of the morphisms that interchange with all others.
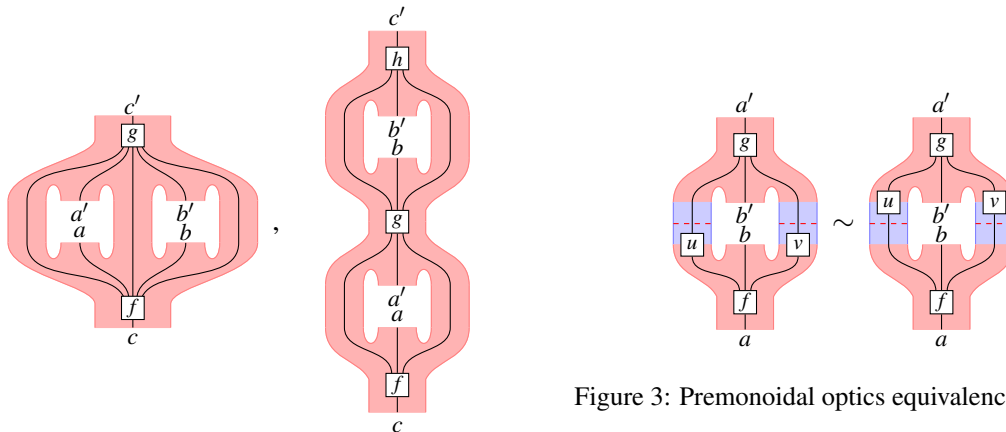

Figure 3: Premonoidal optics equivalence relation.

Figure 2: *Left*: $\otimes_H$, *Right*: $\otimes_V$.

In this article, we will develop the machinery required to deal with optics for premonoidal categories. There is a category $\mathsf{Optic}_{Z\mathscr{C}}(\mathscr{C})$ with objects given by pairs (a, a') of those of $\mathscr{C}$ and homs given by the sets of the form in Figure 3, handled by the coend $\int^{xy \in Z\mathscr{C}} \mathscr{C}(a, x \otimes b \otimes y) \times \mathscr{C}(x \otimes b' \otimes y, a')$. While premonoidal optics can be seen to be a special case of generalised optics for an actegory [15, 40], the full monoidal-like structure of the category has not been discussed before. Optics over a monoidal category $\mathscr{C}$ are equipped with two promonoidal tensor products, $\otimes_H, \otimes_V : \mathsf{Optic}(\mathscr{C}) \times \mathsf{Optic}(\mathscr{C}) \longrightarrow \mathsf{Optic}(\mathscr{C})$, which capture the horizontal and vertical composition of holes (see Figure 2). These two tensors interact to make $\mathsf{Optic}(\mathscr{C})$ into a produoidal category [22].

Over a premonoidal category $\mathscr{C}$ we might hope to equip $\mathsf{Optic}_{Z\mathscr{C}}(\mathscr{C})$ with two tensors analogous to those in Figure 2. While the vertical tensor $\otimes_V$ poses no immediate difficulties, the horizontal tensor $\otimes_H$ does: we cannot expect this to be promonoidal because $\mathscr{C}$ does not satisfy interchange. This requires us to introduce the notion of a *pro-effectful* category which combines the structures of premonoidal and promonoidal categories together – this is our main technical contribution.

We prove that pro-effectful categories are equivalently: *(i)* prostrong promonads; *(ii)* biproactegories (two-sided actions in the category of profunctors) which suitably extend a canonical action on the centre of the category; and *(iii)* pseudomonoids in the bicategory of tight $\mathscr{V}^2$-profunctors.

Each of these gives a different perspective on pro-effectful categories, connecting them, respectively, with monads; the action definition of Freyd-categories given by Levy [33]; the pseudomonoid definition of effectful categories given by Román [43] and the work on closed effectful categories due to Power [36, 37]. In particular, this final perspective demonstrates that pro-effectful categories are equivalent to closed effectful categories on the free tight cocompletion, where the effectful structure is given by a version of Day convolution.

Finally, there is an additional challenge with premonoidal optics: there is the category $\mathsf{Optic}(Z\mathscr{C})$ of optics over the monoidal centre and an embedding, $\mathsf{Optic}(Z\mathscr{C}) \to \mathsf{Optic}_{Z\mathscr{C}}(\mathscr{C})$, of these central optics into the optics over the entire premonoidal category. $\mathsf{Optic}(Z\mathscr{C})$ is equipped with the two promonoidal

structures, $\otimes_H$ and $\otimes_V$, and we would like to understand how these behave in relation to any tensors we can define on $\mathsf{Optic}_{Z\mathscr{C}}(\mathscr{C})$. This requires us to keep track of the centre and understand fully how it behaves in relation to the rest of the premonoidal category.

2 Premonoidal and Effectful Categories

Let us start by formally defining premonoidal categories enriched over a fixed cosmos $\mathscr{V}$, taken to be bicomplete and closed symmetric monoidal. We take some space to spell this out as there are some technicalities involved which do not appear to have been explicitly discussed elsewhere. Unless otherwise indicated "category," "functor," "natural transformation" etc. should be taken to mean $\mathscr{V}$-category etc. We write $\otimes$ for the tensor of $\mathscr{V}$ and for the enriched tensor of categories [31].

Definition 1 (Binoidal Category). A category $\mathscr{C}$ is binoidal when, for each object a, it is equipped with a pair of functors $a \ltimes - : \mathscr{C} \to \mathscr{C}$ and $- \rtimes a : \mathscr{C} \to \mathscr{C}$ such that for all a and b, $a \ltimes b = a \rtimes b$.

In the case of $\mathscr{V} = \mathsf{Set}$, the previous definition is equivalent to the one in terms of the funny tensor product [23, 48]. [1] We now generalize the notion of *central morphism* to the enriched case.

Definition 2 (Centre Piece). Let $\mathscr{C}$ be a binoidal category. A centre piece at objects (a,b) is an object $U(a,b)$ in $\mathscr{V}$, endowed with an arrow $\iota : U(a,b) \to \mathscr{C}(a,b)$, such that for any objects (c,d) the following diagrams commute.

$$
\begin{array}{ccccc}
U(a,b) \otimes \mathscr{C}(c,d) & \xrightarrow{\iota \otimes 1} & \mathscr{C}(a,b) \otimes \mathscr{C}(c,d) & \xrightarrow{(-\rtimes c) \otimes (b \ltimes -)} & \mathscr{C}(a \rtimes c, b \rtimes c) \otimes \mathscr{C}(b \ltimes c, b \ltimes d) \\
{\scriptstyle \iota \otimes 1} \downarrow & & & & \downarrow {\scriptstyle \circ \sigma} \\
\mathscr{C}(a,b) \otimes \mathscr{C}(c,d) & \xrightarrow[(-\rtimes d) \otimes (a \ltimes -)]{} & \mathscr{C}(a \rtimes d, b \rtimes d) \otimes \mathscr{C}(a \ltimes c, a \ltimes d) & \xrightarrow[\circ]{} & \mathscr{C}(a \rtimes c, b \rtimes d)
\end{array}
$$

$$
\begin{array}{ccccc}
\mathscr{C}(c,d) \otimes U(a,b) & \xrightarrow{1 \otimes \iota} & \mathscr{C}(c,d) \otimes \mathscr{C}(a,b) & \xrightarrow{(-\rtimes a) \otimes (d \ltimes -)} & \mathscr{C}(c \rtimes a, d \rtimes a) \otimes \mathscr{C}(d \ltimes a, d \ltimes b) \\
{\scriptstyle 1 \otimes \iota} \downarrow & & & & \downarrow {\scriptstyle \circ \sigma} \\
\mathscr{C}(c,d) \otimes \mathscr{C}(a,b) & \xrightarrow[(-\rtimes b) \otimes (c \ltimes -)]{} & \mathscr{C}(c \rtimes b, d \rtimes b) \otimes \mathscr{C}(c \ltimes a, c \ltimes b) & \xrightarrow[\circ]{} & \mathscr{C}(c \rtimes a, d \rtimes b)
\end{array}
$$

A morphism of centre pieces at (a,b) is an arrow $u : V(a,b) \to U(a,b)$ in $\mathscr{V}$ such that $\iota_U \circ u = \iota_V$. Centre pieces and morphisms of centre pieces form a category $CP(a,b)$.

Definition 3 (Centre). Let $\mathscr{C}$ be binoidal. The centre of $\mathscr{C}$ at objects (a,b) is the universal centre piece, $\iota : Z\mathscr{C}(a,b) \to \mathscr{C}(a,b)$, such that all other centre pieces factorise uniquely through it.

Universal centre pieces assemble to give a category $Z\mathscr{C}$ with the same objects as $\mathscr{C}$ and with hom-objects given by the universal centre pieces. Composition and identities are inherited from $\mathscr{C}$ because (1) and j_a are centre pieces and thus factor via the centre.

Proposition 1. *The arrow* (1), *representing composition inherited from* $\mathscr{C}$, *is a center piece at* (a,c).

$$
Z\mathscr{C}(b,c) \otimes Z\mathscr{C}(a,b) \xrightarrow{\iota \otimes \iota} \mathscr{C}(b,c) \otimes \mathscr{C}(a,b) \xrightarrow{\circ} \mathscr{C}(a,c) \tag{1}
$$

Furthermore, for each object a, the arrow $j_a : I_{\mathscr{V}} \to \mathscr{C}(a,a)$, representing the identities, is a centre piece at (a,a). As a result, $Z\mathscr{C}$ is a category.

---

[1] We avoid that definition in the enriched case because it relies on the discrete category $\mathscr{C}_0$ of objects of a category $\mathscr{C}$ which is ill-defined over arbitrary $\mathscr{V}$.

Proof. In Appendix A.1.   □

Finally we note that the ι assemble to give an identity on objects functor $Z\mathscr{C} \to \mathscr{C}$.

Definition 4 (Central Natural Transformation)**.** Let $\mathscr{C}$ be binoidal and $F, G : \mathscr{D} \to \mathscr{C}$ be two functors. A natural transformation $\eta : F \to G$ is central when the components are central, so that we have a family of morphisms $\eta_a : I_{\mathscr{V}} \to Z\mathscr{C}(Fa, Ga)$ of $\mathscr{V}$ satisfying the naturality diagrams.

Binoidal categories give us the necessary machinery to define premonoidal categories.

Definition 5 (Premonoidal Category)**.** A premonoidal category, $\mathscr{C}$, is a binoidal category endowed with an object i and central natural isomorphisms, $(a \otimes b) \otimes c \cong a \otimes (b \otimes c)$ and $a \otimes i \cong a \cong i \otimes a$, such that the triangle and pentagon equations hold.

Definition 6 (Premonoidal Functor)**.** A premonoidal functor $F : \mathscr{C} \to \mathscr{D}$ between premonoidal categories is a functor which maps central morphisms to central morphisms and which preserves the premonoidal structure up to natural transformations $Fa \otimes Fb \to F(a \otimes b)$ and $i \to Fi$ subject to coherence conditions like those for a monoidal functor. A premonoidal functor is strict when these transformations are identities.

As we have just seen, enriched premonoidal categories are difficult to define - since the coherence isomorphisms need to be central we are required to define binoidal categories first so that we can make sense of this centrality. Even in the case $\mathscr{V} = \mathsf{Set}$, Cat fails to be a monoidal 2-category under the funny tensor product because the funny tensor of natural transformations is not well-defined unless the components are all central [38]. This prevents the swift and elegant definition "*a premonoidal category is a pseudomonoid in* $\mathsf{Cat}_\square$."

Power realised that premonoidal categories are more algebraically well-defined when one shifts to working with premonoidal categories with a specified subcategory of central morphisms [36]. We call these *effectful categories* [43].

Definition 7 (Effectful Category)**.** An effectful category consists of a monoidal category $\mathscr{C}_0$, a premonoidal category $\mathscr{C}_1$ with the same objects as $\mathscr{C}_0$ and a strict, identity on objects, premonoidal functor $J : \mathscr{C}_0 \to \mathscr{C}_1$.

Remark. If $\mathscr{C}_0$ is cartesian, then an effectful category is known as a *Freyd category* [45, 33]. At least one good reason to relax from requiring cartesian structure to arbitrary monoidal structure concerns applications of effectful categories outside of computer science. For instance, there has been interest in effectful structure for models of spacetime and quantum theory where products are not the canonical way of taking joint systems [17, 26], and in the study of Petri nets [5].

Effectful categories can be seen as particular instances of actegories [33] - that is, a category with an action by a monoidal category [6, 12]. An effectful category $J : \mathscr{C}_0 \to \mathscr{C}_1$ specifies a left and right $\mathscr{C}_0$-action on $\mathscr{C}_1$, making $\mathscr{C}_1$ into a $\mathscr{C}_0$-$\mathscr{C}_0$-biactegory. These actions $\ltimes : \mathscr{C}_0 \otimes \mathscr{C}_1 \to \mathscr{C}_1$ and $\rtimes : \mathscr{C}_1 \otimes \mathscr{C}_0 \to \mathscr{C}_1$, are special because they preserve the canonical actions of $\mathscr{C}_0$ on itself, i.e. J extends the action $J\boxtimes = \ltimes(1 \otimes J)$ and $J\boxtimes = \rtimes(J \otimes 1)$, and preserves the coherence isomorphisms.

Effectful categories are also precisely the same thing as strong promonads [27, 24, 43]. Given a strong promonad $T : \mathscr{C} \nrightarrow \mathscr{C}$ there is a canonical premonoidal structure on the Kleisli category Kl_T which on objects acts like the tensor of $\mathscr{C}$. The free functor $F : \mathscr{C} \to \mathsf{Kl}_T$ then constitutes an effectful category. Conversely, given an effectful category $J : \mathscr{C}_0 \to \mathscr{C}_1$, there is a promonad $\mathscr{C}_1(J-, J-) : \mathscr{C}_0 \nrightarrow \mathscr{C}_0$ which can be shown to be strong with strengths induced by the action of $\mathscr{C}_0$ on $\mathscr{C}_1$.

Effectful Categories as Pseudomonoids

In this section, we show that $\mathscr{V}$-enriched effectful categories can be seen as categories enriched in an arrow category $\mathscr{V}^2 = [\to, \mathscr{V}]$, the category of arrows and commutative squares in $\mathscr{V}$ [37].

Proposition 2. *Let $\mathscr{V}$ be a complete, cocomplete, closed symmetric monoidal category. Then $\mathscr{V}^2$ is also a complete, cocomplete, closed symmetric monoidal category and therefore constitutes a cosmos.*

Proof. In Appendix A.2. $\square$

Since $\mathscr{V}^2$ is a cosmos, we can consider categories enriched in $\mathscr{V}^2$ [37]. A $\mathscr{V}^2$-category consists of a pair of categories $\mathscr{C}_0$ and $\mathscr{C}_1$ with the same objects, and an identity on objects functor $J : \mathscr{C}_0 \to \mathscr{C}_1$. A $\mathscr{V}^2$-functor $F : J_{\mathscr{C}} \to J_{\mathscr{D}}$ consists of a pair of functors $F_0 : \mathscr{C}_0 \to \mathscr{D}_0$ and $F_1 : \mathscr{C}_1 \to \mathscr{D}_1$ such that $F_1 J_{\mathscr{C}} = J_{\mathscr{D}} F_0$. A $\mathscr{V}^2$-natural transformation $\eta : F \Rightarrow G$ between $\mathscr{V}^2$-functors $F, G : J_{\mathscr{C}} \to J_{\mathscr{D}}$ consists of natural transformations $\eta^0 : F_0 \Rightarrow G_0$ and $\eta^1 : F_1 \Rightarrow G_1$ with components that satisfy $J_{\mathscr{D}}(\eta^0_c) = \eta^1_c$. When $J_{\mathscr{D}}$ is an embedding, we can think of this transformation simply as having central components, in $\mathscr{D}_0$.

There is a bicategory $\mathscr{V}^2$-Cat of $\mathscr{V}^2$-categories, $\mathscr{V}^2$-functors and $\mathscr{V}^2$-natural transformations. This bicategory has an interesting tensor that arises as a slight modification of the funny tensor product.

Definition 8 (Funny Tensor of $\mathscr{V}^2$-Categories). Given two $\mathscr{V}^2$-categories $J_{\mathscr{C}}$ and $J_{\mathscr{D}}$, their funny tensor $J_{\mathscr{C} \square \mathscr{D}} : \mathscr{C}_0 \otimes \mathscr{D}_0 \to \mathscr{C}_1 \square \mathscr{D}_1$ is the identity on objects functor given by the diagonal of the following pushout in $\mathscr{V}$-Cat.

$$\begin{array}{ccc} \mathscr{C}_0 \otimes \mathscr{D}_0 & \xrightarrow{J_{\mathscr{C}} \otimes 1} & \mathscr{C}_1 \otimes \mathscr{D}_0 \\ {\scriptstyle 1 \otimes J_{\mathscr{D}}} \downarrow & \ulcorner & \downarrow {\scriptstyle i_0} \\ \mathscr{C}_0 \otimes \mathscr{D}_1 & \xrightarrow{i_1} & \mathscr{C}_1 \square \mathscr{D}_1 \end{array} \quad (2)$$

The pushout exists because $\mathscr{V}$ is cocomplete and thus $\mathscr{V}$-Cat is also cocomplete [50]. Given $\mathscr{V}^2$-functors $F : J_{\mathscr{A}} \to J_{\mathscr{B}}$ and $G : J_{\mathscr{C}} \to J_{\mathscr{D}}$ their funny tensor $F \square G$ has components $(F \square G)_0 = F_0 \otimes G_0$ and $(F \square G)_1 = F_1 \square G_1$ given by the unique arrow induced by the pushout. The funny tensor is also well-behaved on $\mathscr{V}^2$-natural transformations because their components $J_{\mathscr{D}}(\eta^0_c) = \eta^1_c$ are central and thus interchange with all other morphisms in $\mathscr{C} \square \mathscr{D}$.

Theorem 1. $\mathscr{V}^2$-Cat *is a monoidal 2-category under the funny tensor $\square$.*

Proof sketch. In Appendix A.3. $\square$

This leads to the main theorem of this section. Theorem 2 is equivalent to the result of Román [43]: effectful categories are pseudomonoids in the bicategory of promonads, promonad homomorphisms and promonad modifications.

Theorem 2. *An effectful category is a pseudomonoid in $\mathscr{V}^2$-Cat$_\square$.*

Proof. In Appendix A.4. $\square$

Theorem 3. *There is an equivalence of bicategories $\mathscr{V}^2$-Cat$_\square \cong \mathscr{V}$-Promonad between the bicategories of $\mathscr{V}^2$-categories under the funny tensor product and the bicategory of promonads.*

Proof sketch. The result follows upon unwinding the definitions in [43] and comparing with those of the present section. $\square$

3   Closed Effectful Categories

In this section, we turn our attention to the closure of effectful categories. Like for monoidal categories, we still want an adjunction between tensoring and the internal-hom, but we only ask that this holds for the centre.

Definition 9 (Closed Effectful Category [36])**.** An effectful category $J : \mathscr{C}_0 \to \mathscr{C}_1$ is right-closed when for each object X, $J(-) \otimes X : \mathscr{C}_0 \to \mathscr{C}_1$ has a right adjoint $[X, -] : \mathscr{C}_1 \to \mathscr{C}_0$. An effectful category is left-closed when for each X, $X \otimes J(-) : \mathscr{C}_0 \to \mathscr{C}_1$ has a right adjoint. We say an effectful category is closed if it is both left and right-closed.

Power proved the following result which generalises Day's result that every monoidal category embeds into a closed monoidal category [19].

Theorem 4 ([36])**.** *Every (small) effectful category embeds into a closed effectful category.*

We say that an effectful category $J : \mathscr{C}_0 \to \mathscr{C}_1$ is *small* when both $\mathscr{C}_0$ and $\mathscr{C}_1$ are small. One might expect that the closed effectful category into which small J embeds would be given by the free $\mathscr{V}^2$-cocompletion $[J^{\mathrm{op}}, \mathscr{V}^2]$. To the contrary, this category does not have a non-trivial closed effectful structure. Power showed that the correct category to consider is given by the free *tight* cocompletion - that is, the cocompletion in only $\mathscr{V}$-colimits, not all weighted colimits. In the case of $\mathscr{V} = \mathsf{Set}$ these are precisely the "conical" colimits.

Theorem 5 ([37])**.** *The free tight cocompletion of a small $\mathscr{V}^2$-category $J : \mathscr{C}_0 \to \mathscr{C}_1$ is the bijective on objects functor $\mathrm{Lan}^L_{J^{op}} : \widehat{\mathscr{C}_0} \to \overline{\mathscr{C}_1}$ induced by the functor of $\mathrm{Lan}_{J^{op}} : \widehat{\mathscr{C}_0} \to \widehat{\mathscr{C}_1}$, via its canonical factorisation into a bijective on objects functor followed by a fully faithful functor (its bo-ff factorisation, see Appendix A.5).*

Remark. As a consequence, the following diagram commutes, giving a factorisation (y^L, y^R) of the Yoneda embedding $y : J \to [J^{\mathrm{op}}, \mathscr{V}^2]$, via the free $\mathscr{V}$-cocompletion.

The category $\overline{\mathscr{C}_1} := \mathrm{Im}(\mathrm{Lan}_{J^{op}})$ has as objects, presheaves $F : \mathscr{C}_0^{\mathrm{op}} \to \mathscr{V}$. The hom $\overline{\mathscr{C}_1}(F, G)$, is given by $\widehat{\mathscr{C}_1}(\mathrm{Lan}_{J^{op}} F, \mathrm{Lan}_{J^{op}} G)$. The category $\overline{\mathscr{C}_1}$ has another description.

Proposition 3. *Let $J : \mathscr{C}_0 \to \mathscr{C}_1$ be an effectful category with corresponding strong promonad $T(-,-) := \mathscr{C}_1(J-, J-)$. Let $\overline{\mathscr{C}_1} = \mathrm{Im}(Lan_{J^{op}})$ be the category given by the bo-ff factorisation of $\mathrm{Lan}_{J^{op}}$. Then $\overline{\mathscr{C}_1}$ is isomorphic to the Kleisli category $\mathsf{Kl}_{\widehat{T}}$ of the monad $\widehat{T} : \widehat{\mathscr{C}_0} \to \widehat{\mathscr{C}_0}$ induced by T on the presheaf category.*

Proof. The result follows by the adjunction between extension and restriction of presheaves along J^{op}. It is spelt out in Appendix A.6.   □

We can describe the effectful structure on $\mathrm{Lan}^L_{J^{op}}$ as follows. The centre $\widehat{\mathscr{C}_0}$ is closed monoidal under Day convolution while the premonoidal structure on $\overline{\mathscr{C}_1}$ is given by the unique arrow induced by the

universal property of the coend. That is, given $\eta : \mathrm{Lan}_{J^{\mathrm{op}}}F \Rightarrow \mathrm{Lan}_{J^{\mathrm{op}}}G$ in $\overline{\mathscr{C}_1}$ there is a unique arrow giving the left side of the premonoidal structure (and similarly for the right side).

$$
\begin{array}{ccc}
\vcenter{\hbox{\includegraphics{diag1}}} & \overset{\eta}{\Rightarrow} & \vcenter{\hbox{\includegraphics{diag2}}}
\end{array}
\tag{3}
$$

We see that $\mathrm{Lan}_{J^{\mathrm{op}}}^{L}$ factors through the centre $Z(\overline{\mathscr{C}_1})$ of $\overline{\mathscr{C}_1}$, for if we are given $\eta : F \Rightarrow G$ and $\varepsilon : H \Rightarrow K$ there is a unique arrow making it clear that the interchange law holds.

$$
\vcenter{\hbox{\includegraphics{diag3}}} \overset{\eta \otimes \varepsilon}{\Longrightarrow} \vcenter{\hbox{\includegraphics{diag4}}}
$$

The internal-hom $[F, -] : \overline{\mathscr{C}_1} \to \widehat{\mathscr{C}_0}$ is given by the following presheaf which can be checked to be right adjoint to $\mathrm{Lan}_{J^{\mathrm{op}}}^{L}(-) \otimes F$ by coend calculus (see Appendix A.7).

$$
[F, G](-) \cong \int_{c \in \mathscr{C}_1} \mathscr{V}\big((\mathrm{Lan}_{J^{\mathrm{op}}}F)(c), (\mathrm{Lan}_{J^{\mathrm{op}}}G)(J - \otimes c) \big)
$$

4 $\mathscr{V}^2$-Profunctors

Let us now study the structure of $\mathscr{V}^2$-profunctors $P : J_{\mathscr{D}}^{\mathrm{op}} \otimes J_{\mathscr{C}} \to \mathscr{V}^2$. By the following result we are able to unpack P into a pair of $\mathscr{V}$-profunctors together with a natural transformation between them. The $\mathscr{V}^2$-natural transformations $\phi : P \Rightarrow Q$ can also be similarly unpacked.

Proposition 4. *Let $P : J_{\mathscr{D}}^{op} \otimes J_{\mathscr{C}} \to \mathscr{V}^2$ be a $\mathscr{V}^2$-profunctor. Then P is a triple of:*

1. *a $\mathscr{V}$-profunctor $P_0 : \mathscr{D}_0^{op} \otimes \mathscr{C}_0 \to \mathscr{V}$,*

2. *a $\mathscr{V}$-profunctor $P_1 : \mathscr{D}_1^{op} \otimes \mathscr{C}_1 \to \mathscr{V}$,*

3. *a $\mathscr{V}$-natural transformation $\eta : P_0 \Rightarrow P_1(J^{op} \otimes J)$.*

A $\mathscr{V}^2$-natural transformation $\phi : P \Rightarrow Q$ consists of $\mathscr{V}$-natural transformations $\phi_0 : P_0 \Rightarrow Q_0$ and $\phi_1 : P_1 \Rightarrow Q_1$ such that $(\phi_1(J^{op} \otimes J))\eta^P = \eta^Q \phi_0$.

Proof. This follows by applying a $\mathscr{V}$-enriched version of a result by Power [37, Prop. 24] to the functor category $[J_{\mathscr{D}}^{\mathrm{op}} \otimes J_{\mathscr{C}}, \mathscr{V}^2] \cong \mathsf{Prof}(J_{\mathscr{C}}, J_{\mathscr{D}})$. $\qquad\square$

The next proposition demonstrates that the coend of a $\mathscr{V}^2$-profunctor P is given by the coends of P_0 and P_1 together with a canonical arrow between them.

Proposition 5. *Let $P : J^{op} \otimes J \to \mathscr{V}^2$ be a $\mathscr{V}^2$-endoprofunctor. Then the coend $\int^c P(c, c)$ is given by the arrow $\int^c P_0(c, c) \to \int^c P_1(c, c)$ induced by η and the adjunction $y_J \dashv y^J$ in $\mathscr{V}$-Prof.*

Proof. In Appendix A.8. $\qquad\square$

$\mathscr{V}^2$-endoprofunctors and the $\mathscr{V}^2$-natural transformations assemble into a $\mathscr{V}^2$-category $[J^{\mathrm{op}} \otimes J, \mathscr{V}^2] \cong$ $\mathsf{Prof}(J,J)$. The category $\mathsf{Prof}(J,J)_0$ consists of the $\mathscr{V}^2$-profunctors and $\mathscr{V}^2$-natural transformations as outlined in Proposition 4, while $\mathsf{Prof}(J,J)_1$ has homs consisting of only the components ϕ_1 of the natural transformations. The identity on objects functor $\mathsf{Prof}(J,J)_0 \to \mathsf{Prof}(J,J)_1$ forgets the ϕ_0 components.

As with any other category of endoprofunctors $\mathsf{Prof}(J,J)$ has a closed monoidal structure given by composition of the profunctors. Given $P = (P_0, P_1, \eta_P)$ and $Q = (Q_0, Q_1, \eta_Q)$, their composition is given by $QP = (Q_0 P_0, Q_1 P_1, \eta_{QP})$ - we compose the underlying profunctors and take η_{QP} to be given by

$$\int^c Q(-,c) \otimes P(c,-) \xrightarrow{\int \eta_Q \otimes \eta_P} \int^{c \in \mathscr{C}_0} Q(J-,Jc) \otimes P(Jc,J-) \xrightarrow{y_J \dashv y^J} \int^{c \in \mathscr{C}_1} Q(J-,c) \otimes P(c,J-).$$

4.1   Tight Profunctors

In Section 3 we saw that effectful structure on $J : \mathscr{C}_0 \to \mathscr{C}_1$ induced a closed effectful structure on the free tight cocompletion of J. It turns out that this effectful structure on J is only a sufficient and not necessary condition for closed effectful structure on the free tight cocompletion of J. Analogously to the case of monoidal categories where, in order for the presheaf category $\widehat{\mathscr{C}}$ to be closed monoidal it is only necessary that the category $\mathscr{C}$ is promonoidal [19, 18], we only require J to be a "pro-effectful" category. To define these categories we need firstly to study the class of profunctors which factor through the tight cocompletion. This will be the aim of this section.

To define pro-effectful categories we would like to replace the functors of a effectful category with profunctors, but we have a problem: we cannot consider arbitrary $\mathscr{V}^2$-profunctors $P : J_{\mathscr{D}}^{\mathrm{op}} \otimes J_{\mathscr{C}} \to \mathscr{V}^2$ because these assign arbitrary presheaves $J_{\mathscr{D}}^{\mathrm{op}} \to \mathscr{V}^2$ to objects of $J_{\mathscr{C}}$. These presheaves will not in general be contained in the free tight cocompletion. Thus, we need a restricted class of profunctors, those that we call the *tight* profunctors.

Definition 10 (Tight $\mathscr{V}^2$-Profunctor)**.** A tight $\mathscr{V}^2$-profunctor $P : J_{\mathscr{C}} \longrightarrow J_{\mathscr{D}}$ is a $\mathscr{V}^2$-functor $P : J_{\mathscr{C}} \to \overline{J_{\mathscr{D}}}$, where $\overline{J_{\mathscr{D}}} \cong \mathsf{Lan}_{J_{\mathscr{D}}^{\mathrm{op}}}^L$ is the free tight cocompletion of $J_{\mathscr{D}}$.

Similarly to how a profunctor $P : \mathscr{C} \longrightarrow \mathscr{D}$ is equivalently a cocontinuous functor between free cocompletions $\widehat{P} : \widehat{\mathscr{C}} \to \widehat{\mathscr{D}}$, tight $\mathscr{V}^2$-profunctors are *tightly* cocontinuous functors between free tight cocompletions.

Definition 11 (Tightly Cocontinuous Functor)**.** A $\mathscr{V}^2$-functor $F : J_{\mathscr{C}} \to J_{\mathscr{D}}$ between tightly cocomplete categories is tightly cocontinuous if it preserves all tight colimits.

Theorem 6 ([31])**.** *Let $\overline{J_{\mathscr{C}}}$ be the closure of $J_{\mathscr{C}}$ in $[J_{\mathscr{C}}^{\mathrm{op}}, \mathscr{V}^2]$ under tight colimits and write $y^L : J_{\mathscr{C}} \to \overline{J_{\mathscr{C}}}$ for the inclusion. Then for tightly cocomplete $J_{\mathscr{D}}$, there is an equivalence*

$$Lan_{y^L} : [J_{\mathscr{C}}, J_{\mathscr{D}}] \cong \mathsf{Cocont}_{tight}(\overline{J_{\mathscr{C}}}, J_{\mathscr{D}})$$

where the right-hand is the category of tightly cocontinuous functors. This exhibits $\overline{J_{\mathscr{C}}}$ as the free tight cocompletion of $J_{\mathscr{C}}$.

Indeed, y^L is fully faithful (see Appendix A.9), so that there is a natural isomorphism $F \cong (\mathsf{Lan}_{y^L} F) y^L$. Consequently, we can think of a tight $\mathscr{V}^2$-profunctor $P : J_{\mathscr{C}} \to \overline{J_{\mathscr{D}}}$ as a tightly cocontinuous functor $\tilde{P} : \overline{J_{\mathscr{C}}} \to \overline{J_{\mathscr{D}}}$. We can now define the following bicategory of tight $\mathscr{V}^2$-profunctors.

Definition 12. Denote by $\mathscr{V}^2\text{-}\mathsf{Prof}^{\mathsf{Tight}}$ the bicategory that has

- 0-cells the $\mathscr{V}^2$-categories $J : \mathscr{C}_0 \to \mathscr{C}_1$,

- 1-cells, $P : J_\mathscr{C} \longrightarrow J_\mathscr{D}$, the tight $\mathscr{V}^2$-profunctors $P : J_\mathscr{C} \to \overline{J_\mathscr{D}}$,

- 2-cells the $\mathscr{V}^2$-natural transformations.

Composition of 1-cells is given by taking the left Kan extension along y^L and composing the functors we obtain $Q \circ P = (\mathrm{Lan}_{y^L} Q) P$.

Remark. We could also have defined tight $\mathscr{V}^2$-profunctors $J_\mathscr{C} \to \overline{J_\mathscr{D}}$ as usual $\mathscr{V}^2$-profunctors $J_\mathscr{C} \to [J_\mathscr{D}^{\mathrm{op}}, \mathscr{V}^2]$ that factorise via the embedding $y^R : \overline{J_\mathscr{D}} \to [J_\mathscr{D}^{\mathrm{op}}, \mathscr{V}^2]$. Their usual composition as profunctors coincides (up to natural isomorphism) with the composition defined previously because y^R is fully faithful (see Appendix A.9) and thus the unit of the Kan extension along y^R is an isomorphism, $F \cong (\mathrm{Lan}_{y^R} F) y^R$. It follows that

$$Q \circ P = (\mathrm{Lan}_y Q) P = (\mathrm{Lan}_{y^R y^L} Q) P \cong (\mathrm{Lan}_{y^R} \mathrm{Lan}_{y^L} Q) P = (\mathrm{Lan}_{y^R} \mathrm{Lan}_{y^L} Q) y^R P' \cong (\mathrm{Lan}_{y^L} Q) P'.$$

Proposition 6 (External Tensor Product). *Let $J_\mathscr{C}$ and $J_\mathscr{D}$ be $\mathscr{V}^2$-categories and write $\overline{J_\mathscr{C}}$ and $\overline{J_\mathscr{D}}$ be their free tight cocompletions. Then there is a $\mathscr{V}^2$-functor*

$$\hat{\otimes} : \overline{J_\mathscr{C}} \,\square\, \overline{J_\mathscr{D}} \to \overline{J_{\mathscr{C} \square \mathscr{D}}} \tag{4}$$

with components that act on objects as $(F \hat{\otimes} G)(c,d) := Fc \otimes Gd$.

Proof. In Appendix A.10. $\square$

$\mathscr{V}^2$-$\mathsf{Prof}^{\mathsf{Tight}}$ has an interesting tensor product given by generalising the funny tensor product.

Definition 13 (Funny Tensor Product of Tight $\mathscr{V}^2$-Profunctors). On categories the funny tensor acts like in $\mathscr{V}^2$-Cat. On tight $\mathscr{V}^2$-profunctors $P : J_\mathscr{A} \to \overline{J_\mathscr{B}}$ and $Q : J_\mathscr{C} \to \overline{J_\mathscr{D}}$ we define their funny tensor to be given by their funny tensor in $\mathscr{V}^2$-Cat composed with the external tensor of free tight cocompletions (4):

$$J_\mathscr{A} \,\square\, J_\mathscr{C} \xrightarrow{P \square Q} \overline{J_\mathscr{B}} \,\square\, \overline{J_\mathscr{D}} \xrightarrow{\hat{\otimes}} \overline{J_{\mathscr{B} \square \mathscr{D}}}$$

Theorem 7. $\mathscr{V}^2$-$\mathsf{Prof}^{\mathsf{Tight}}$ *is a monoidal bicategory under the funny tensor product.*

5   Pro-effectful Categories

Definition 14. A pro-effectful category is a pseudomonoid in $\mathscr{V}^2$-$\mathsf{Prof}_\square^{\mathsf{Tight}}$. Explicitly, a pro-effectful category $J_\mathscr{C}$ is a $\mathscr{V}^2$-category equipped with

- a tensor product tight $\mathscr{V}^2$-profunctor $P : J_{\mathscr{C} \square \mathscr{C}} \longrightarrow J_\mathscr{C}$,

- and a unit tight $\mathscr{V}^2$-profunctor $I : 1 \longrightarrow J_\mathscr{C}$,

together with $\mathscr{V}^2$-natural isomorphisms $P(P \,\square\, 1) \overset{\alpha}{\cong} P(1 \,\square\, P)$ and $P(I \,\square\, 1) \overset{\lambda}{\cong} 1 \overset{\rho}{\cong} P(1 \,\square\, I)$ such that the triangle and pentagon equations hold.

Like their effectful counterparts, pro-effectful categories also have an "actegorical definition" - they are a particular instance of a category equipped with an action by a promonoidal category. This requires us firstly to weaken actegories to proactegories.

Definition 15 (Proactegory)**.** A left proactegory is a promonoidal category $(\mathscr{C}_0, P, I)$ and a category $\mathscr{C}_1$ equipped with a left proaction by $\mathscr{C}_0$, that is, a profunctor $L : \mathscr{C}_0 \otimes \mathscr{C}_1 \nrightarrow \mathscr{C}_1$ and natural isomorphisms

$$\int^{X \in \mathscr{C}_1} L(A,B,X) \otimes L(X,C,D) \overset{a}{\cong} \int^{X \in \mathscr{C}_0} L(A,X,D) \otimes P(X,B,C), \quad \int^{X \in \mathscr{C}_0} L(A,X,B) \otimes I(X) \overset{l}{\cong} \mathscr{C}_1(A,B),$$

satisfying similar coherence diagrams as for an actegory. A biproactegory is simultaneously a left and right proactegory with an additional natural isomorphism

$$\int^X R(D,X,C) \otimes L(X,A,B) \overset{b}{\cong} \int^X L(D,A,X) \otimes R(X,B,C)$$

satisfying similar coherences as for a biactegory.

The following result generalises the equivalence between effectful categories and certain actegories [33] to the pro-effectful case.

Proposition 7. *A pro-effectful category is equivalently the following data:*

- *a promonoidal category $(\mathscr{C}_0, P_0, I_0)$,*

- *a category $\mathscr{C}_1$ with the same objects as $\mathscr{C}_0$ and an identity on objects functor $J : \mathscr{C}_0 \to \mathscr{C}_1$,*

- *left and right $\mathscr{C}_0$-proactions on $\mathscr{C}_1$, $P_1^L : \mathscr{C}_0 \otimes \mathscr{C}_1 \nrightarrow \mathscr{C}_1$ and $P_1^R : \mathscr{C}_1 \otimes \mathscr{C}_0 \nrightarrow \mathscr{C}_1$, which extend the canonical proactions of $\mathscr{C}_0$ on itself:*

$$
\begin{array}{ccc}
\mathscr{C}_0 \otimes \mathscr{C}_0 & \xrightarrow{\;P_0\;} & \mathscr{C}_0 \\
{\scriptstyle 1 \otimes yJ}\big\downarrow & & \big\downarrow{\scriptstyle yJ} \\
\mathscr{C}_0 \otimes \mathscr{C}_1 & \xrightarrow[\;P_1^L\;]{} & \mathscr{C}_1
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{C}_0 \otimes \mathscr{C}_0 & \xrightarrow{\;P_0\;} & \mathscr{C}_0 \\
{\scriptstyle yJ \otimes 1}\big\downarrow & & \big\downarrow{\scriptstyle yJ} \\
\mathscr{C}_1 \otimes \mathscr{C}_0 & \xrightarrow[\;P_1^R\;]{} & \mathscr{C}_1
\end{array}
\qquad (5)
$$

- *a natural isomorphism $P_1^R(P_1^L \otimes 1) \cong P_1^L(1 \otimes P_1^R)$ making $\mathscr{C}_1$ into a $\mathscr{C}_0$-$\mathscr{C}_0$-biproactegory.*

Proof. In Appendix A.11. □

The next proposition generalises the equivalence between effectful categories and strong promonads [27, 24, 43] to the pro-effectful case. The proof methods are related to those for promonoidal monads in [20].

Proposition 8. *A pro-effectful category is equivalently a prostrong promonad (see Definition 20).*

Proof. In Appendix A.12. □

Pro-effectful categories are also exactly what is required to place a closed effectful structure on the free tight cocompletion of a $\mathscr{V}^2$-category. This generalises Day's theorem [19, 18] from monoidal to effectful categories, thus also generalising the result of Power on closed effectful embeddings of effectful categories [36, 37]. The result follows by generalising the methods of Day's original proof, and from the folklore results regarding Day convolution for actegories, see [28, 11].

Theorem 8. *There is an equivalence between pro-effectful structures on J and closed effectful structures on the free tight cocompletion $\overline{J} = Lan_{J^{op}}^L$.*

Proof. In Appendix A.13. □

Finally, we note some connections between pro-effectful categories and the premulticategories of Staton and Levy [45], which generalise multicategories by dropping the interchange law. Just as how promonoidal categories are examples of (co)multicategories [21], pro-effectful categories are examples of (co)premulticategories. Given a pro-effectful category $J : \mathscr{C}_0 \to \mathscr{C}_1$, there is a co-premulticategory $\mathfrak{C}$ with objects given by those of $\mathscr{C}_1$. For $a, b \in \mathfrak{C}$ the class of arrows is given by $\mathfrak{C}(a;b) := \mathscr{C}_1(a,b)$ and for $a, b, c \in \mathfrak{C}$ the class of arrows is given by $\mathfrak{C}(a;b,c) := P_1(a,b,c)$. The rest of the classes of arrows are defined inductively.

It is worth noting that there exist examples of pro-effectful categories which provide non-degenerate examples of premulticategories where the interchange law does not hold (in contrast to promonoidal and monoidal categories which are multicategories) and where the "tensor" is not representable (in contrast to monoidal and premonoidal categories). For instance, the premonoidal optics introduced in the next section are an example of such a category.

6 Premonoidal Optics

In a seminal work on optics, Riley [40] introduced the notion of "effectful optics": optics over the Kleisli category of a strong monad. These optics allow the emergence of side-effects, and extend the optics of pure functional programming to other programming languages with effects; with a similar purpose, Abou-Saleh et al [1] have introduced "monadic lenses". More recently, much applied category theory has been written about optics that create effects in different categories [8, 10, 15, 44].

We introduce a novel definition of optic over an effectful category that justifies this previous terminology: optics over the Kleisli category of a strong monad are particular cases of our effectful optics. We also introduce a proeffectful algebra over them that had been previously neglected. In this section we will present the category of optics over a premonoidal category and outline its two tensor-like structures, analogous to those in Figure 2.

Suppose we fix a premonoidal category $\mathscr{C}$ and write $J : Z\mathscr{C} \to \mathscr{C}$ for the inclusion of the centre. There is a $\mathscr{V}^2$-category $\mathsf{Optic}(J)$ with objects given by pairs $\mathbf{a} := (a, a')$ of those of J, i.e. pairs of those of the underlying premonoidal category $\mathscr{C}$. The homs are given by $\int^{xy} Z\mathscr{C}(a, x \otimes b \otimes y) \otimes Z\mathscr{C}(x \otimes b' \otimes y, a') \to \int^{xy \in Z\mathscr{C}} \mathscr{C}(a, x \otimes b \otimes y) \otimes \mathscr{C}(x \otimes b' \otimes y, a')$, as in Figure 4. Thus $\mathsf{Optic}(J)_0 = \mathsf{Optic}(Z\mathscr{C})$ is the usual category of optics over the centre and $\mathsf{Optic}(J)_1 = \mathsf{Optic}_{Z\mathscr{C}}(\mathscr{C})$ is the category of optics given by the action of the centre $Z\mathscr{C}$ on the whole premonoidal category $\mathscr{C}$. The identity on objects functor $\mathsf{Optic}(Z\mathscr{C}) \to \mathsf{Optic}_{Z\mathscr{C}}(\mathscr{C})$ is the one induced by J.

Theorem 9. $\mathsf{Optic}(J)$ *is a promonoidal* $\mathscr{V}^2$*-category. The* $\mathscr{V}^2$*-profunctors forming the tensor product* $P : \mathsf{Optic}(J) \otimes \mathsf{Optic}(J) \rightarrowtail \mathsf{Optic}(J)$ *and unit* $I : 1 \rightarrowtail \mathsf{Optic}(J)$ *have components given in Figures 5 and 6. These are explicitly,*

$$P_0(\mathbf{c}, \mathbf{a}, \mathbf{b}) = \int^{xx'yy'} Z\mathscr{C}(c, x \otimes a \otimes x') \otimes Z\mathscr{C}(x \otimes a' \otimes x', y \otimes b \otimes y') \otimes Z\mathscr{C}(y \otimes b' \otimes y', c'),$$

$$P_1(\mathbf{c}, \mathbf{a}, \mathbf{b}) = \int^{xx'yy' \in Z\mathscr{C}} \mathscr{C}(c, x \otimes a \otimes x') \otimes \mathscr{C}(x \otimes a' \otimes x', y \otimes b \otimes y') \otimes \mathscr{C}(y \otimes b' \otimes y', c'),$$

$$I_0(\mathbf{a}) = Z\mathscr{C}(a, a'), \qquad I_1(\mathbf{a}) = \mathscr{C}(a, a').$$
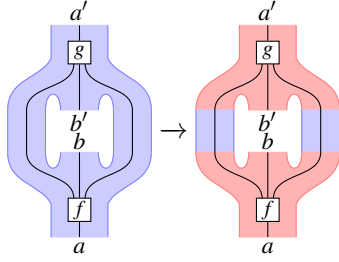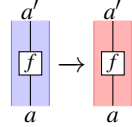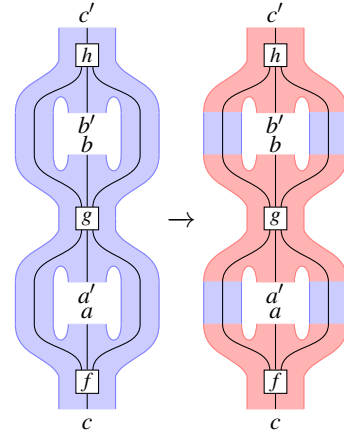
Proof. In Appendix A.14. □

Figure 4: Homs.

Figure 5: Promonoidal unit I.

Figure 6: Promonoidal tensor P.

Now let us turn our attention to another tensor-like structure on $\mathsf{Optic}_{1_{Z\mathscr{C}}}(J)$, this one induced by the premonoidal structure on $\mathscr{C}$.

Theorem 10. $\mathsf{Optic}(J)$ *is a pro-effectful category. The tight $\mathscr{V}^2$-profunctors forming the tensor product* $P : \mathsf{Optic}(J) \otimes \mathsf{Optic}(J) \to \overline{\mathsf{Optic}(J)}$ *and unit* $I : 1 \to \overline{\mathsf{Optic}(J)}$ *have components which act on objects as,*

$$P_0(\mathbf{c}, \mathbf{a}, \mathbf{b}) = P_1(\mathbf{c}, \mathbf{a}, \mathbf{b}) = \int^{xyz} Z\mathscr{C}(c, x \otimes a \otimes y \otimes b \otimes z) \otimes Z\mathscr{C}(x \otimes a' \otimes y \otimes b' \otimes z, c'),$$

$$I_0(\mathbf{a}) = I_1(\mathbf{a}) = Z\mathscr{C}(a, a'). \tag{6}$$

Proof. In Appendix A.15. □

On the homs of $Z\mathscr{C}$, P_0 and I_0 act in the expected way, essentially by nesting of optics. On the homs of $\mathscr{C}$, P_1 and I_1 act somewhat unusually. Formally non-central optics are sent to natural transformations between left Kan extensions of the expressions in (6), that is between presheaves of the form:

$$(\mathrm{Lan}_{J^{\mathrm{op}} \otimes J} P_0)(\mathbf{c}, \mathbf{a}, \mathbf{b}) \cong \int^{wvxyz} \mathscr{C}(c, Jw) \otimes Z\mathscr{C}(w, x \otimes a \otimes y \otimes b \otimes z) \otimes Z\mathscr{C}(x \otimes a' \otimes y \otimes b' \otimes z, v) \otimes \mathscr{C}(Jv, c')$$

$$\cong \int^{xyz \in Z\mathscr{C}} \mathscr{C}(c, x \otimes a \otimes y \otimes b \otimes z) \otimes \mathscr{C}(x \otimes a' \otimes y \otimes b' \otimes z, c')$$

$$(\mathrm{Lan}_{J^{\mathrm{op}} \otimes J} I_0)(\mathbf{a}) \cong \int^{xy} \mathscr{C}(a, Jx) \otimes Z\mathscr{C}(x, y) \otimes \mathscr{C}(Jy, a') \cong \int^{x \in Z\mathscr{C}} \mathscr{C}(a, x) \otimes \mathscr{C}(x, a')$$

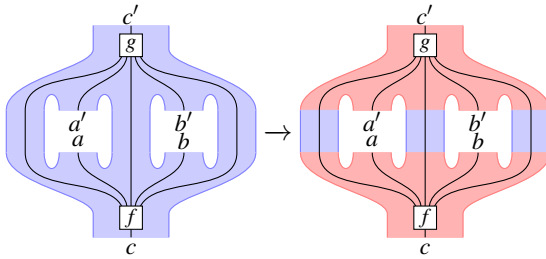This justifies thinking of the pro-effectful structure as having the components described in Figures 7 and 8.
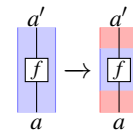

Figure 7: Pro-effectful tensor.

Figure 8: Pro-effectful unit.

Acknowledgements

References

[1] Faris Abou-Saleh, James Cheney, Jeremy Gibbons, James McKinna & Perdita Stevens (2016): *Reflections on Monadic Lenses*. In: *A List of Successes That Can Change the World - Essays Dedicated to Philip Wadler on the Occasion of His 60th Birthday*, Lecture Notes in Computer Science 9600, Springer, pp. 1–31, doi:10.1007/978-3-319-30936-1_1.

[2] Samson Abramsky & Bob Coecke (2004): *A categorical semantics of quantum protocols*. In: *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, 2004*, pp. 415–425, doi:10.1109/LICS.2004.1319636.

[3] Kazuyuki Asada (2010): *Arrows Are Strong Monads*. In: *Proceedings of the Third ACM SIGPLAN Workshop on Mathematically Structured Functional Programming*, MSFP '10, Association for Computing Machinery, New York, NY, USA, p. 33–42, doi:10.1145/1863597.1863607.

[4] Andrea Asperti & Guiseppe Longo (1991): *Categories, Types, and Structures*. MIT Press.

[5] John C. Baez, John Foley & Joe Moeller (2019): *Network Models from Petri Nets with Catalysts*. Compositionality 1, doi:10.32408/compositionality-1-4.

[6] Jean Bénabou (1967): *Introduction to bicategories*. In: *Reports of the Midwest Category Seminar*, Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 1–77, doi:10.1007/BFb0074299.

[7] Guillaume Boisseau (2020): *String Diagrams for Optics*. In Zena M. Ariola, editor: *5th International Conference on Formal Structures for Computation and Deduction (FSCD 2020)*, Leibniz International Proceedings in Informatics (LIPIcs) 167, Schloss Dagstuhl–Leibniz-Zentrum für Informatik, Dagstuhl, Germany, pp. 17:1–17:18, doi:10.4230/LIPIcs.FSCD.2020.17.

[8] Joe Bolt, Jules Hedges & Philipp Zahn (2019): *Bayesian open games*, doi:10.48550/arXiv.1910.03656. arXiv:1910.03656.

[9] Filippo Bonchi, Robin Piedeleu, Pawel Sobociński & Fabio Zanasi (2019): *Graphical affine algebra*. In: *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, IEEE, pp. 1–12, doi:10.1109/LICS.2019.8785877.

[10] Dylan Braithwaite & Jules Hedges (2022): *Dependent Bayesian Lenses: Categories of Bidirectional Markov Kernels with Canonical Bayesian Inversion*, doi:10.48550/arXiv.2209.14728. arXiv:2209.14728.

[11] Alexander Campbell (2018): *Skew-Enriched Categories*. Applied Categorical Structures 26, p. 597–615, doi:10.1007/s10485-017-9504-0.

[12] Matteo Capucci & Bruno Gavranović (2022): *Actegories for the Working Amthematician*, doi:10.48550/arXiv.2203.16351. arXiv:2203.16351.

[13] Giulio Chiribella, Giacomo Mauro D'Ariano & Paolo Perinotti (2008): *Quantum Circuit Architecture*. *Physical Review Letters* 101(6), doi:10.1103/physrevlett.101.060401.

[14] Giulio Chiribella, Giacomo Mauro D'Ariano & Paolo Perinotti (2008): *Transforming quantum operations: Quantum supermaps*. *EPL (Europhysics Letters)* 83(3), p. 30004, doi:10.1209/0295-5075/83/30004.

[15] Bryce Clarke, Derek Elkins, Jeremy Gibbons, Fosco Loregian, Bartosz Milewski, Emily Pillmore & Mario Román (2020): *Profunctor optics, a categorical update*, doi:10.48550/arXiv.2001.07488. arXiv:2001.07488.

[16] Bob Coecke & Aleks Kissinger (2017): *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*. Cambridge University Press, doi:10.1017/9781316219317.

[17] Marc Comeau (2012): *Premonoidal *-Categories and Algebraic Quantum Field Theory*. Ph.D. thesis, University of Ottawa, doi:10.20381/ruor-5524.

[18] Brian Day (1970): *Construction of Biclosed Categories*. Ph.D. thesis, University of New South Wales, doi:10.26190/unsworks/8048.

[19] Brian Day (1970): *On closed categories of functors*. In: *Reports of the Midwest Category Seminar IV*, 137, Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 1–38, doi:10.1007/BFb0060438.

[20] Brian Day (1977): *Note on monoidal monads*. *Journal of the Australian Mathematical Society* 23(3), p. 292–311, doi:10.1017/S1446788700018929.

[21] Brian Day, Elango Panchadcharam & Ross Street (2005): *On centres and lax centres for promonoidal categories*. In: *Colloque International Charles Ehresmann: 100 ans*.

[22] Matt Earnshaw, James Hefford & Mario Román (2023): *The Produoidal Algebra of Process Decomposition*, doi:10.48550/ARXIV.2301.11867. arXiv:2301.11867.

[23] François Foltz, Christian Lair & Gregory M. Kelly (1980): *Algebraic categories with few monoidal biclosed structures or none*. *Journal of Pure and Applied Algebra* 17, pp. 171–177, doi:10.1016/0022-4049(80)90082-1.

[24] Richard Garner & Ignacio López Franco (2016): *Commutativity*. *Journal of Pure and Applied Algebra* 220(5), pp. 1707–1751, doi:10.1016/j.jpaa.2015.09.003.

[25] James Hefford & Cole Comfort (2022): *Coend Optics for Quantum Combs*, doi:10.48550/ARXIV.2205.09027. arXiv:2205.09027.

[26] James Hefford & Aleks Kissinger (2022): *On the pre- and promonoidal structure of spacetime*, doi:10.48550/ARXIV.2206.09678. arXiv:2206.09678.

[27] Bart Jacobs, Chris Heunen & Ichiro Hasuo (2009): *Categorical semantics for arrows*. *Journal of Functional Programming* 19(3-4), p. 403–438, doi:10.1017/S0956796809007308.

[28] G. Janelidze & G.M. Kelly (2001): *A Note On Actions of a Monoidal Category*. *Theory and Applications of Categories* 9(4), p. 61–91.

[29] Alan Jeffrey (1997): *Premonoidal categories and flow graphs*. In: *Second Workshop on Higher-Order Operational Techniques in Semantics, HOOTS 1997, Stanford, CA, USA, December 8-12, 1997, Electronic Notes in Theoretical Computer Science* 10, Elsevier, p. 51, doi:10.1016/S1571-0661(05)80688-7.

[30] André Joyal & Ross Street (1991): *The geometry of tensor calculus, I*. Advances in Mathematics 88(1), pp. 55–112, doi:10.1016/0001-8708(91)90003-P.

[31] Gregory Kelly (1982): *Basic Concepts of Enriched Category Theory*. Cambridge University Press.

[32] Aleks Kissinger & Sander Uijlen (2019): *A categorical semantics for causal structure*. Logical Methods in Computer Science, doi:10.23638/LMCS-15(3:15)2019.

[33] Paul Levy (2003): *Call-By-Push-Value*. Springer Dordrecht, doi:10.1007/978-94-007-0954-6.

[34] Fosco Loregian (2021): *(Co)end Calculus*. London Mathematical Society Lecture Note Series, Cambridge University Press, doi:10.1017/9781108778657.

[35] Craig Pastro & Ross Street (2008): *Doubles for Monoidal Categories*. Theory and Applications of Categories 21(4), pp. 61–75.

[36] John Power (2002): *Premonoidal categories as categories with algebraic structure*. Theoretical Computer Science 278(1), pp. 303–321, doi:10.1016/S0304-3975(00)00340-6. Mathematical Foundations of Programming Semantics 1996.

[37] John Power (2006): *Generic models for computational effects*. Theoretical Computer Science 364(2), pp. 254–269, doi:10.1016/j.tcs.2006.08.006.

[38] John Power & Edmund Robinson (1997): *Premonoidal categories and notions of computation*. Mathematical Structures in Computer Science 7(5), p. 453–468, doi:10.1017/S0960129597002375.

[39] John Power & Hayo Thielecke (2002): *Closed Freyd-and κ-categories*. In: *Automata, Languages and Programming: 26th International Colloquium, ICALP'99 Prague, Czech Republic, July 11–15, 1999 Proceedings*, Springer, pp. 625–634.

[40] Mitchell Riley (2018): *Categories of Optics*, doi:10.48550/arXiv.1809.00738. arXiv:1809.00738.

[41] Mario Román (2020): *Profunctor optics and traversals*, doi:10.48550/arXiv.2001.08045. arXiv:2001.08045.

[42] Mario Román (2021): *Open Diagrams via Coend Calculus*. Electronic Proceedings in Theoretical Computer Science 333, p. 65–78, doi:10.4204/eptcs.333.5.

[43] Mario Román (2022): *Promonads and String Diagrams for Effectful Categories*, doi:10.48550/ARXIV.2205.07664. arXiv:2205.07664.

[44] David I. Spivak (2019): *Generalized Lens Categories via functors $\mathscr{C}^{\mathrm{op}} \to$ Cat*, doi:10.48550/arXiv.1908.02202. arXiv:1908.02202.

[45] Sam Staton & Paul Blain Levy (2013): *Universal Properties of Impure Programming Languages*. SIGPLAN Not. 48(1), p. 179–192, doi:10.1145/2480359.2429091.

[46] Ross Street (1972): *The formal theory of monads*. Journal of Pure and Applied Algebra 2(2), pp. 149–168, doi:10.1016/0022-4049(72)90019-9.

[47] Daisuke Tambara (2006): *Distributors on a tensor category*. Hokkaido Mathematical Journal 35(2), pp. 379 – 425, doi:10.14492/hokmj/1285766362.

[48] Mark Weber (2013): *Free Products of Higher Operad Algebras*. Theory and Applications of Categories 28(2).

[49] Matt Wilson, Giulio Chiribella & Aleks Kissinger (2022): *Quantum Supermaps are Characterized by Locality*, doi:10.48550/ARXIV.2205.09844. arXiv:2205.09844.

[50] Harvey Wolff (1974): *V-cat and V-graph*. Journal of Pure and Applied Algebra 4(2), pp. 123–135, doi:10.1016/0022-4049(74)90018-8.
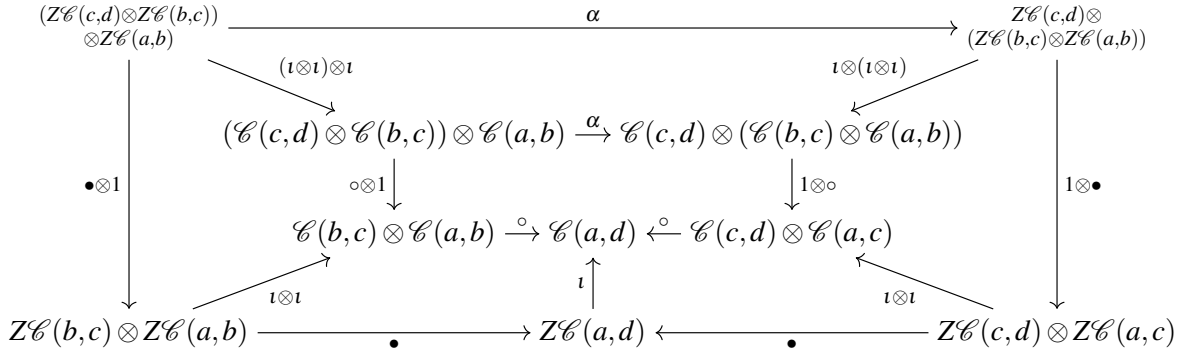
A Proofs

A.1 Proof of Proposition 1

Proof. Ignoring associativity isomorphisms, the following diagram commutes for any *x* and *y*, showing that the arrow (1) is a centre piece at (a,c).
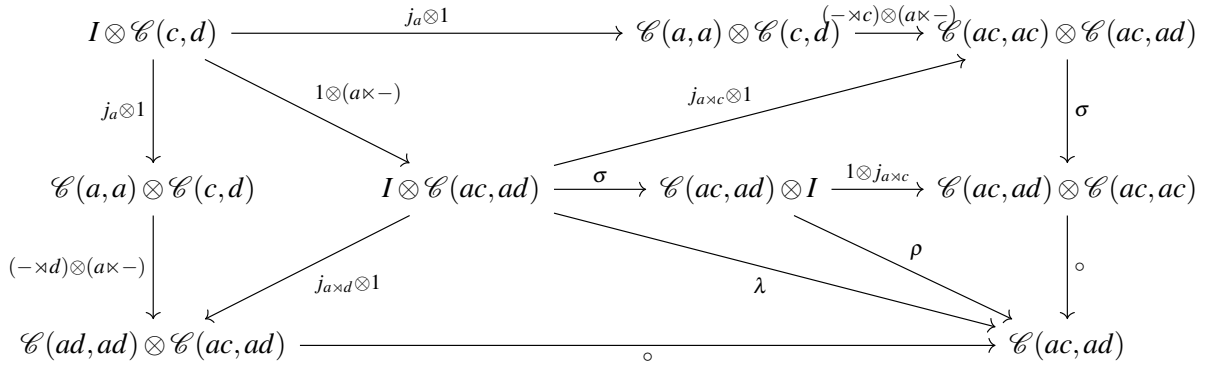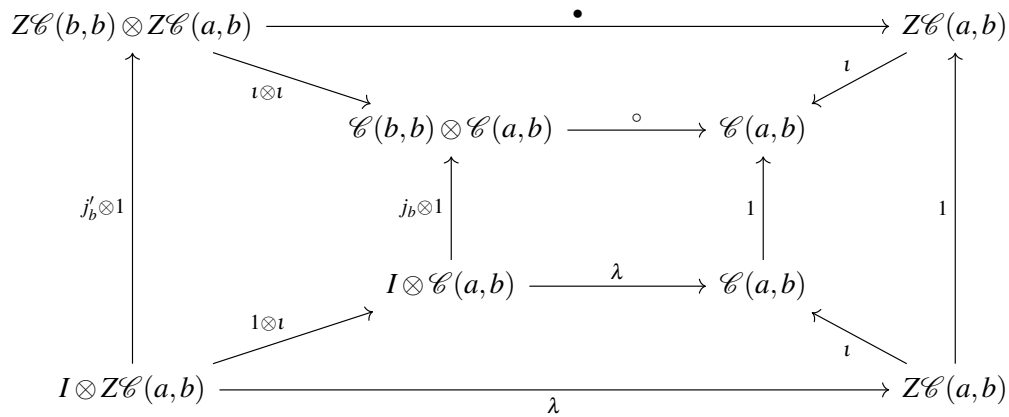
An analogous diagram also commutes for the other interchange law. As a result $\circ(\iota \otimes \iota)$ factorises uniquely via the universal centre piece. This gives us a composition operation for $Z\mathscr{C}$, say $\bullet$, such that $\circ(\iota \otimes \iota) = \iota\bullet$. Composition in $Z\mathscr{C}$ is associative because the following diagram commutes.

$$
\begin{array}{ccc}
(Z\mathscr{C}(c,d)\otimes Z\mathscr{C}(b,c))\otimes Z\mathscr{C}(a,b) & \xrightarrow{\;\;\alpha\;\;} & Z\mathscr{C}(c,d)\otimes(Z\mathscr{C}(b,c)\otimes Z\mathscr{C}(a,b))
\end{array}
$$

with inner square

$$
(\mathscr{C}(c,d)\otimes\mathscr{C}(b,c))\otimes\mathscr{C}(a,b) \xrightarrow{\;\alpha\;} \mathscr{C}(c,d)\otimes(\mathscr{C}(b,c)\otimes\mathscr{C}(a,b))
$$

morphisms: $(\iota\otimes\iota)\otimes\iota$, $\iota\otimes(\iota\otimes\iota)$, $\bullet\otimes1$, $\circ\otimes1$, $1\otimes\circ$, $1\otimes\bullet$

$$
\mathscr{C}(b,c)\otimes\mathscr{C}(a,b) \xrightarrow{\;\circ\;} \mathscr{C}(a,d) \xleftarrow{\;\circ\;} \mathscr{C}(c,d)\otimes\mathscr{C}(a,c)
$$

$$
Z\mathscr{C}(b,c)\otimes Z\mathscr{C}(a,b) \xrightarrow{\;\bullet\;} Z\mathscr{C}(a,d) \xleftarrow{\;\bullet\;} Z\mathscr{C}(c,d)\otimes Z\mathscr{C}(a,c)
$$

(with $\iota\otimes\iota$ and ι connecting maps)

The following diagram commutes for all c,d, showing that j_a is a centre piece at (a,a).

$$
I\otimes\mathscr{C}(c,d) \xrightarrow{\;j_a\otimes1\;} \mathscr{C}(a,a)\otimes\mathscr{C}(c,d) \xrightarrow{(-\ltimes c)\otimes(a\ltimes-)} \mathscr{C}(ac,ac)\otimes\mathscr{C}(ac,ad)
$$

with maps $j_a\otimes1$, $1\otimes(a\ltimes-)$, $j_{a\times c}\otimes1$, σ, $\mathscr{C}(a,a)\otimes\mathscr{C}(c,d)$, $(-\ltimes d)\otimes(a\ltimes-)$, $I\otimes\mathscr{C}(ac,ad) \xrightarrow{\sigma} \mathscr{C}(ac,ad)\otimes I \xrightarrow{1\otimes j_{a\times c}} \mathscr{C}(ac,ad)\otimes\mathscr{C}(ac,ac)$, $j_{a\times d}\otimes1$, λ, ρ, $\circ$

$$
\mathscr{C}(ad,ad)\otimes\mathscr{C}(ac,ad) \xrightarrow{\;\circ\;} \mathscr{C}(ac,ad)
$$

A similar diagram commutes for the other interchange law. As a result j_a factorises uniquely via the universal centre piece as $j_a = \iota j'_a$. Finally, note that the following diagram commutes showing the left unit law holds for composition in $Z\mathscr{C}$.

$$
Z\mathscr{C}(b,b)\otimes Z\mathscr{C}(a,b) \xrightarrow{\;\bullet\;} Z\mathscr{C}(a,b)
$$

with maps: $\iota\otimes\iota$, ι, $\mathscr{C}(b,b)\otimes\mathscr{C}(a,b) \xrightarrow{\circ} \mathscr{C}(a,b)$, $j'_b\otimes1$, $j_b\otimes1$, 1, 1, $I\otimes\mathscr{C}(a,b) \xrightarrow{\lambda} \mathscr{C}(a,b)$, $1\otimes\iota$, ι, $I\otimes Z\mathscr{C}(a,b) \xrightarrow{\lambda} Z\mathscr{C}(a,b)$

The right unit law is very similar. $\qquad\square$

A.2   Proof of Proposition 2

Proof. The category $\mathcal{V}^2$ inherits a symmetric monoidal structure from $\mathcal{V}$. On objects, which are arrows of $\mathcal{V}$, this monoidal structure acts by

$$(a_0 \xrightarrow{f} a_1) \otimes (b_0 \xrightarrow{g} b_1) := a_0 \otimes b_0 \xrightarrow{f \otimes g} a_1 \otimes b_1.$$

On morphisms, which are squares of $\mathcal{V}$, it acts in an analogous way.

Now, consider three objects of $\mathcal{V}^2$, say $a_0 \xrightarrow{f} a_1$, $b_0 \xrightarrow{g} b_1$ and $c_0 \xrightarrow{h} c_1$. We aim to construct the internal-hom and demonstrate the natural isomorphism $\mathcal{V}^2(f \otimes g, h) \cong \mathcal{V}^2(f, [g,h])$. Consider the following pullback in $\mathcal{V}$, which exists because $\mathcal{V}$ is complete.

$$
\begin{array}{ccc}
\mathcal{V}(b_0,c_0) \times_{\mathcal{V}(b_0,c_1)} \mathcal{V}(b_1,c_1) & \xrightarrow{\ p_0\ } & \mathcal{V}(b_0,c_0) \\
{\scriptstyle p_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathcal{V}(1,h)} \\
\mathcal{V}(b_1,c_1) & \xrightarrow[\ \mathcal{V}(g,1)\ ]{} & \mathcal{V}(b_0,c_1)
\end{array}
$$

Let us now demonstrate that the internal-hom is given by the projection out of the pullback, $[g,h] = p_1$. To give an arrow $z : f \to p_1$ of $\mathcal{V}^2$ is to give a pair of arrows such that the following square commutes,

$$
\begin{array}{ccc}
a_0 & \xrightarrow{\quad\quad f \quad\quad} & a_1 \\
{\scriptstyle z_0}\downarrow & & \downarrow{\scriptstyle z_1} \\
\mathcal{V}(b_0,c_0) \times_{\mathcal{V}(b_0,c_1)} \mathcal{V}(b_1,c_1) & \xrightarrow[\ p_1\ ]{} & \mathcal{V}(b_1,c_1)
\end{array}
\tag{7}
$$

To give z_0 is to give $z_{00} := p_0 z_0 : a_0 \to \mathcal{V}(b_0,c_0)$ and $z_{01} := p_1 z_0 : a_0 \to \mathcal{V}(b_1,c_1)$ such that $\mathcal{V}(1,h)z_{00} = \mathcal{V}(g,1)z_{01}$. To make (7) commute is to ask $z_{01} = z_1 f$.

Under the adjunction due to the closure of $\mathcal{V}$, we now have $z_{00}^* : a_0 \otimes b_0 \to c_0$ and $z_{01}^* : a_0 \otimes b_1 \to c_1$ such that $h z_{00}^* = z_{01}^*(1 \otimes g)$. Also from z_1 we get a $z_1^* : a_1 \otimes b_1 \to c_1$ such that $z_1^*(f \otimes 1) = z_{01}^*$. As a result we find $h z_{00}^* = z_1^*(f \otimes g)$, so that z is equivalent to giving $z^* : f \otimes g \to h$.

Finally, the completeness and cocompleteness of $\mathcal{V}^2$ are inherited from $\mathcal{V}$, pointwise. □

A.3   Proof of Theorem 1

Proof sketch. The behaviour of the funny tensor on functors is encapsulated by the following cube.

Functoriality of $\square$ on 1-cells follows by pasting of cubes and the uniqueness of the arrows induced by the pushout.

Explicitly, we have $(\alpha : F \Rightarrow F') \square (\beta : G \Rightarrow G')$ has components $(\alpha \square \beta)_{cd}^0 = (\alpha_c^0, \beta_d^0)$ and $(\alpha \square \beta)_{cd}^1 = (\alpha_c^1, \beta_d^1) = (J\alpha_c^0, J\beta_d^0)$. Naturality of this transformation follows from naturality of α and β and from the centrality of the components. □

A.4 Proof of Theorem 2

Proof. A pseudomonoid in $\mathscr{V}^2$-$\mathsf{Cat}_\square$ consists of a $\mathscr{V}^2$-category $J : \mathscr{C}_0 \to \mathscr{C}_1$ equipped with $\mathscr{V}^2$-functors $\boxtimes : J \square J \to J$ and $I : 1 \to J$, such that there are $\mathscr{V}^2$-natural isomorphisms

$$\boxtimes(\boxtimes \otimes 1) \stackrel{\alpha}{\cong} \boxtimes(1 \otimes \boxtimes) \text{ and } \boxtimes(I \otimes 1) \stackrel{\lambda}{\cong} 1 \stackrel{\rho}{\cong} \boxtimes(1 \otimes I).$$

Note that $\boxtimes$ consists of two functors $\boxtimes_0 : \mathscr{C}_0 \otimes \mathscr{C}_0 \to \mathscr{C}_0$ and $\boxtimes_1 : \mathscr{C}_1 \square \mathscr{C}_1 \to \mathscr{C}_1$ such that $J\boxtimes_0 = \otimes_1 J_{\mathscr{C} \square \mathscr{C}}$. $\boxtimes_0$ together with I_0 and the natural isomorphisms α_0, ρ_0 and λ_0, give a monoidal structure on $\mathscr{C}_0$.

The $\mathscr{C}_0$-biaction on $\mathscr{C}_1$ is given by the compositions $\ltimes := \boxtimes i_1$ and $\rtimes := \boxtimes i_0$. That J preserves the canonical actions given by $\otimes_0$ on $\mathscr{C}_0$ follows by the diagram (2) and the equality $J\otimes_0 = \otimes_1 J_{\mathscr{C} \square \mathscr{C}}$, together with the fact that α_1, ρ_1 and λ_1 have components in the image of J. The coherence equations of the biaction are a consequence of those of α_1, ρ_1 and λ_1: for instance α_1 is a natural isomorphism between functors with type $\mathscr{C}_1 \square \mathscr{C}_1 \square \mathscr{C}_1 \to \mathscr{C}_1$. This amounts to "separate" naturality in each $\mathscr{C}_1$ of the domain which in turns induces the left, bimodule and right coherences for the biaction. $\qquad\square$

A.5 Bijective on Objects-Fully Faithful (bo-ff) Factorisations

Proposition 9. *Any $\mathscr{V}$-functor $F : \mathscr{C} \to \mathscr{D}$ factorises as the composition of a bijective-on-objects functor and a fully-faithful functor.*

Proof. Define $\mathrm{Im}F$ to be the $\mathscr{V}$-category with objects given by those of $\mathscr{C}$ and hom-objects given by $\mathrm{Im}F(a,b) := \mathscr{D}(Fa,Fb)$. Its composition and identities are inherited from $\mathscr{D}$. The $\mathscr{V}$-functor F now factorises as the composition of $F^L : \mathscr{C} \to \mathrm{Im}F$ and $F^R : \mathrm{Im}F \to \mathscr{D}$, where the $\mathscr{V}$-functor F^L is identity on objects and on homs, $F_{ab}^L : \mathscr{C}(a,b) \to \mathrm{Im}F(a,b)$ is given by F_{ab}; and the $\mathscr{V}$-functor F^R acts as F on objects, and on homs, $F_{ab}^R : \mathrm{Im}F(a,b) \to \mathscr{D}(Fa,Fb)$ is given by $1_{\mathscr{D}(Fa,Fb)}$. $\qquad\square$

A.6 Proof of Proposition 3

Proof. The promonad $T : \mathscr{C}_0 \nrightarrow \mathscr{C}_0$ induces a monad $\hat{T} : \hat{\mathscr{C}}_0 \to \hat{\mathscr{C}}_0$ given by the cocontinuous functor corresponding to the profunctor T. By definition, the Kleisli category $\mathsf{Kl}_{\hat{T}}$ has as objects presheaves $F : \mathscr{C}_0^{\mathrm{op}} \to \mathscr{V}$ while the hom $\mathsf{Kl}_{\hat{T}}(F,G)$ is given by $\hat{\mathscr{C}}_0(F,\hat{T}G)$.

We see that

$$(\hat{T}G)(-) \cong \int^x T(-,x) \otimes Gx = \int^x \mathscr{C}_1(J-,Jx) \otimes Gx \cong (\mathrm{Lan}_{J^{\mathrm{op}}} G)(J-)$$

There is an adjunction between extension and restriction of presheaves:

$$\begin{aligned}
\mathrm{Nat}(F,\hat{T}G) &\cong \mathrm{Nat}(F,(\mathrm{Lan}_{J^{\mathrm{op}}} G)(J-)) \cong \int_x \mathscr{V}\left(Fx,(\mathrm{Lan}_{J^{\mathrm{op}}} G)(Jx)\right) \\
&\cong \int_x \mathscr{V}\left(Fx, \int_y \mathscr{V}\left(\mathscr{C}_1(y,Jx),(\mathrm{Lan}_{J^{\mathrm{op}}} G)(y)\right)\right) \\
&\cong \int_{xy} \mathscr{V}\left(Fx \otimes \mathscr{C}_1(y,Jx),(\mathrm{Lan}_{J^{\mathrm{op}}} G)(y)\right) \\
&\cong \int_y \mathscr{V}\left(\int^x \mathscr{C}_1(y,Jx) \otimes Fx,(\mathrm{Lan}_{J^{\mathrm{op}}} G)(y)\right) \\
&\cong \mathrm{Nat}(\mathrm{Lan}_{J^{\mathrm{op}}} F, \mathrm{Lan}_{J^{\mathrm{op}}} G)
\end{aligned}$$

Thus to give a natural transformation $F \Rightarrow \hat{T}G$ is equivalent to giving one $\mathrm{Lan}_{J^{\mathrm{op}}}F \Rightarrow \mathrm{Lan}_{J^{\mathrm{op}}}G$. This demonstrates that $\overline{\mathscr{C}_1} \cong \mathrm{Kl}_{\hat{T}}$ $\qquad\qquad\square$

A.7   Proof of Closure of $\mathrm{Lan}_{J^{\mathrm{op}}}^L$

Proof.

$$
\begin{aligned}
\widehat{\mathscr{C}_0}(F,[G,H]) &\cong \int_{c \in \mathscr{C}_0} \mathscr{V}\left(Fc,[G,H](c)\right) \\
&\cong \int_{c \in \mathscr{C}_0, d \in \mathscr{C}_1} \mathscr{V}\left(Fc, \mathscr{V}\left((\mathrm{Lan}_{J^{\mathrm{op}}}G)(d),(\mathrm{Lan}_{J^{\mathrm{op}}}H)(Jc \otimes d)\right)\right) \\
&\cong \int_{c \in \mathscr{C}_0, d \in \mathscr{C}_1} \mathscr{V}\left(Fc \otimes (\mathrm{Lan}_{J^{\mathrm{op}}}G)(d),(\mathrm{Lan}_{J^{\mathrm{op}}}H)(Jc \otimes d)\right) \\
&\cong \int_{c \in \mathscr{C}_0, d, d' \in \mathscr{C}_1} \mathscr{V}\left(Fc \otimes (\mathrm{Lan}_{J^{\mathrm{op}}}G)(d) \otimes \mathscr{C}_1(d', Jc \otimes d),(\mathrm{Lan}_{J^{\mathrm{op}}}H)(d')\right) \\
&\cong \int_{d' \in \mathscr{C}_1} \mathscr{V}\left(\int^{c,c' \in \mathscr{C}_0, d \in \mathscr{C}_1} Fc \otimes Gc' \otimes \mathscr{C}_1(d, Jc') \otimes \mathscr{C}_1(d', Jc \otimes d),(\mathrm{Lan}_{J^{\mathrm{op}}}H)(d')\right) \\
&\cong \int_{d' \in \mathscr{C}_1} \mathscr{V}\left((\mathrm{Lan}_{J^{\mathrm{op}}}(F \star G))(d'),(\mathrm{Lan}_{J^{\mathrm{op}}}H)(d')\right) \\
&\cong \overline{\mathscr{C}_1}(F \otimes G, H)
\end{aligned}
$$

Left-closure of $\mathrm{Lan}_{J^{\mathrm{op}}}^L$ follows similarly. $\square$

A.8   Proof of Proposition 5

Proof. Suppose we have a $\mathscr{V}^2$-extranatural family $w_c : P(c,c) \to d$. Then we have the following commutative diagram:

In particular, the families $w_c^0 : P_0(c,c) \to d_0$ and $w_c^1 : P_1(c,c) \to d_1$ are $\mathscr{V}$-extranatural and thus factorise via their respective coends giving arrows $\int^c P_0(c,c) \to d_0$ and $\int^c P_1(c,c) \to d_1$ making the obvious diagrams commute. Now note that the arrows $P_0(c,c) \xrightarrow{\eta_{cc}} P_1(c,c) \xrightarrow{\mathrm{copr}_c} \int^c P_1(c,c)$ are $\mathscr{V}$-extranatural, this induces a arrow $\int^c P_0(c,c) \to \int^c P_1(c,c)$. $\square$

A.9 Proof y^L and y^R are Fully-Faithful

Proof. Fix $J : \mathscr{C}_0 \to \mathscr{C}_1$. We start with $y^L : J \to \bar{J}$ which acts as:

$$c \mapsto \mathscr{C}_0(-, c)$$

$$(f : c \to c') \in \mathscr{C}_0 \overset{y_0^L}{\mapsto} \mathscr{C}_0(-, f) \in \widehat{\mathscr{C}_0}$$

$$(f : c \to c') \in \mathscr{C}_1 \overset{y_1^L}{\mapsto} \mathscr{C}_1(-, f) \in \overline{\mathscr{C}_1}$$

by the Yoneda lemma it is clear that this is both full and faithful.

Now consider $y^R : \bar{J} \to \widehat{J}$ which acts as:

$$(F : \mathscr{C}_0 \to \mathscr{V}) \mapsto (F, \mathrm{Lan}_{J^{\mathrm{op}}} F, \varepsilon : F \Rightarrow (\mathrm{Lan}_{J^{\mathrm{op}}} F) J)$$

$$(\eta : F \Rightarrow G) \in \widehat{\mathscr{C}_0} \overset{y_0^R}{\mapsto} (\eta, \mathrm{Lan}_{J^{\mathrm{op}}} \eta)$$

$$(\eta : F \Rightarrow G) \in \overline{\mathscr{C}_1} \overset{y_1^R}{\mapsto} \eta$$

It is clear to see that y_1^R is fully-faithful because the hom of $[J^{\mathrm{op}}, \mathscr{V}^2]_1$ is precisely given by the natural transformations between left Kan extensions, $\mathrm{Lan}_{J^{\mathrm{op}}} F \Rightarrow \mathrm{Lan}_{J^{\mathrm{op}}} G$. To see that y_0^R is fully-faithful we note that an arrow $(F, \mathrm{Lan}_{J^{\mathrm{op}}} F, \varepsilon^F) \to (G, \mathrm{Lan}_{J^{\mathrm{op}}} G, \varepsilon^G)$ is given by a pair of natural transformations $\eta_0 : F \Rightarrow \eta_1$ and $\eta_1 : \mathrm{Lan}_{J^{\mathrm{op}}} F \Rightarrow \mathrm{Lan}_{J^{\mathrm{op}}} G$ such that $(\eta_1 J) \varepsilon^F = \varepsilon^G \eta_0$. Once η_0 is chosen, then the η_1 satisfying this equality is uniquely determined by the universal property of the left Kan extension. $\square$

A.10 Proof of Proposition 6

Proof. To give (4) is to give a pair of functors such that the following square commutes:

$$
\begin{array}{ccc}
\widehat{\mathscr{C}_0} \otimes \widehat{\mathscr{D}_0} & \longrightarrow & \overline{\mathscr{C}_1} \,\square\, \overline{\mathscr{D}_1} \\
{\scriptstyle \hat{\otimes}_0}\downarrow & & \downarrow{\scriptstyle \otimes_1} \\
\widehat{\mathscr{C}_0 \otimes \mathscr{D}_0} & \xrightarrow[\mathrm{Lan}^L_{J^{\mathrm{op}}_{\mathscr{C}\square\mathscr{D}}}]{} & \overline{\mathscr{C}_1} \,\square\, \overline{\mathscr{D}_1}
\end{array}
$$

The tensor $\hat{\otimes}_0$ acts on objects following the formula $(F \hat{\otimes} G)(c, d) := Fc \otimes Gd$ and on morphisms in the obvious way. The tensor $\hat{\otimes}_1$ also acts following the same formula, note that a morphism of $\overline{\mathscr{C}_1} \,\square\, \overline{\mathscr{D}_1}$ is a free composition of natural transformations $\alpha : \mathrm{Lan}_{J^{\mathrm{op}}_{\mathscr{C}}} F \Rightarrow \mathrm{Lan}_{J^{\mathrm{op}}_{\mathscr{C}}} F'$ and $\beta : \mathrm{Lan}_{J^{\mathrm{op}}_{\mathscr{D}}} G \Rightarrow \mathrm{Lan}_{J^{\mathrm{op}}_{\mathscr{D}}} G'$ with $(1; \beta)(\alpha; 1) \neq (\alpha; 1)(1; \beta)$ in general. Each such arrow induces a natural transformation $\mathrm{Lan}_{J^{\mathrm{op}}_{\mathscr{C}\square\mathscr{D}}} (F \otimes G) \Rightarrow \mathrm{Lan}_{J^{\mathrm{op}}_{\mathscr{C}\square\mathscr{D}}} (F' \otimes G')$, for instance:

$$\mathrm{Lan}_{J^{\mathrm{op}}_{\mathscr{C}\square\mathscr{D}}} (F \otimes G) \cong \mathrm{Lan}_{i_1^{\mathrm{op}}} (\mathrm{Lan}_{J^{\mathrm{op}}_{\mathscr{C}}} F \otimes G)$$

$$\xrightarrow{\mathrm{Lan}_{i_1^{\mathrm{op}}} (\alpha \otimes 1)} \mathrm{Lan}_{i_1^{\mathrm{op}}} (\mathrm{Lan}_{J^{\mathrm{op}}_{\mathscr{C}}} F' \otimes G)$$

$$\cong \mathrm{Lan}_{i_0^{\mathrm{op}}} (F' \otimes \mathrm{Lan}_{J^{\mathrm{op}}_{\mathscr{D}}} G)$$

$$\xrightarrow{\mathrm{Lan}_{i_0^{\mathrm{op}}} (1 \otimes \beta)} \mathrm{Lan}_{i_0^{\mathrm{op}}} (F' \otimes \mathrm{Lan}_{J^{\mathrm{op}}_{\mathscr{D}}} G')$$

$$\cong \mathrm{Lan}_{J^{\mathrm{op}}_{\mathscr{C}\square\mathscr{D}}} (F' \otimes G')$$

$\square$

A.11 Proof of Proposition 7

Proof. Fix a pro-effectful category (J,P,I). J is a $\mathcal{V}^2$-category so we have two categories $\mathscr{C}_0$ and $\mathscr{C}_1$ with the same objects and an identity on objects functor $J : \mathscr{C}_0 \to \mathscr{C}_1$.

The tight $\mathcal{V}^2$-profunctor $P : J_{\mathscr{C}\square\mathscr{C}} \nrightarrow J_{\mathscr{C}}$ consists of a profunctor $P_0 : \mathscr{C}_0 \otimes \mathscr{C}_0 \nrightarrow \mathscr{C}_0$ and a functor $P_1 : \mathscr{C}_1 \square \mathscr{C}_1 \to \overline{\mathscr{C}_1}$. Similarly, the tight $\mathcal{V}^2$-profunctor $I : 1 \nrightarrow J_{\mathscr{C}}$ consists of presheaves $I_0 : \mathscr{C}_0^{\mathrm{op}} \to \mathcal{V}$ and $I_1 = \mathrm{Lan}_{J^{\mathrm{op}}} I_0 : \mathscr{C}_1^{\mathrm{op}} \to \mathcal{V}$. (P_0, I_0) induce a promonoidal structure on $\mathscr{C}_0$.

P_1 induces the left and right proactions of $\mathscr{C}_0$ on $\mathscr{C}_1$. Starting with the left proaction, P_1 induces a functor $y_1^R P_1 i_1 =: P_1^L : \mathscr{C}_0 \otimes \mathscr{C}_1 \to \widehat{\mathscr{C}_1}$. It follows that:

$$P_1^L(1 \otimes J) = y_1^R P_1 i_1 (1 \otimes J) = y_1^R P_1 J_{\mathscr{C}\square\mathscr{C}} = \mathrm{Lan}_{J^{\mathrm{op}} \otimes 1 \otimes 1} P_0$$

showing that (5) commutes and that the left proaction extends the canonical one on $\mathscr{C}_0$. A similar argument holds for the right proaction.

Suppose now that we start with the data specified in the proposition. The equalities (5) together with the universal property of the pushout induce a functor $P_1 : \mathscr{C}_1 \square \mathscr{C}_1 \to \overline{\mathscr{C}_1}$

and it follows that $P_1 J_{\mathscr{C}\square\mathscr{C}} = \mathrm{Lan}_{J^{\mathrm{op}}}^L P_0$ making (P_0, P_1) the components of a tight $\mathcal{V}^2$-profunctor $P : J_{\mathscr{C}\square\mathscr{C}} \nrightarrow J_{\mathscr{C}}$. The presheaf $I_0 : \mathscr{C}_0^{\mathrm{op}} \to \mathcal{V}$ together with its Kan extension $I_1 := \mathrm{Lan}_{J^{\mathrm{op}}} I_0$ give the components of a $\mathcal{V}^2$-profunctor $I : 1 \nrightarrow J$. Checking all the coherences is a long but ultimately routine calculation. $\square$

A.12 Proof of Proposition 8

Proof. Take a prostrong promonad $T : \mathscr{C} \nrightarrow \mathscr{C}$. We will show we have the data of Proposition 7.

T has a Kleisli category in $\mathcal{V}$-Prof and there is an identity on objects free functor $F : \mathscr{C} \to \mathsf{Kl}_T$. By assumption $\mathscr{C}$ has a promonoidal structure (P_0, I_0) and we can use the left and right prostrengths to define left and right proactions of $\mathscr{C}$ on Kl_T. On objects the left proaction acts as $P_1^L(-, c, Fc') := \int^x \mathsf{Kl}_T(-, Fx) \otimes P_0(x, c, c')$ extending the canonical proaction on the centre, so that (5) commutes. Its action on homs is induced by the strength $\int^c P_0(-, -, c) \otimes T(c, -) \Rightarrow \int^c T(-, c) \otimes P_0(c, -, -)$.

Conversely, suppose we are given a pro-effectful category $J : \mathscr{C}_0 \to \mathscr{C}_1$. Then $T(-,=) := \mathscr{C}_1(J-, J=)$ a promonad on $\mathscr{C}_0$ where the promonad multiplication and units are given by composition in $\mathscr{C}_1$. Moreover, $\mathscr{C}_1$ is precisely the Kleisli category of T. Now, since J is pro-effectful, $\mathscr{C}_0$ is promonoidal and we are left to show that T is prostrong over this structure. By Proposition 7, we have left and right proactions of $\mathscr{C}_1$ on $\mathscr{C}_0$ which preserve the canonical proaction on the centre:

From these we can construct the prostrength of T, for instance the left prostrength is given as follows.

$\square$
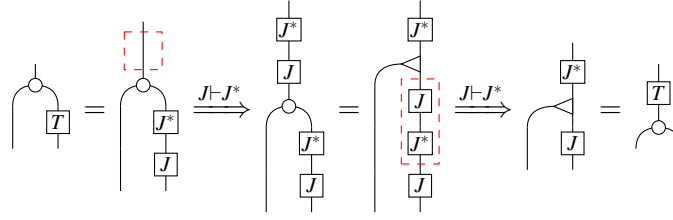
A.13 Proof of Theorem 8

Proof. Suppose $J : \mathscr{C}_0 \to \mathscr{C}_1$ is a pro-effectful category. We will show that $\mathrm{Lan}^L_{J^{\mathrm{op}}} : \widehat{\mathscr{C}_0} \to \overline{\mathscr{C}_1}$ is a closed premonoidal category. Since $\mathscr{C}_0$ is promonoidal, $\widehat{\mathscr{C}_0}$ is closed monoidal under Day convolution.

As for the premonoidal structure on $\overline{\mathscr{C}_1}$: on objects it is the same as on $\widehat{\mathscr{C}_0}$. On morphisms, suppose we are given a $\eta : F \Rightarrow G$ in $\overline{\mathscr{C}_1}$. Then we have a $\eta : \mathrm{Lan}_{J^{\mathrm{op}}} F \Rightarrow \mathrm{Lan}_{J^{\mathrm{op}}} G$ and we can describe the left hand part of the premonoidal structure by

$$
\begin{aligned}
\mathrm{Lan}_{J^{\mathrm{op}}}(F \star F')(-) &\cong \int^{abc} \mathscr{C}_1(-, Jc) \otimes P_0(c, a, b) \otimes Fa \otimes F'b \\
&\cong \int^{ab} P_1^R(-, Ja, b) \otimes Fa \otimes F'b \\
&\cong \int^{bc} P_1^R(-, c, b) \otimes (\mathrm{Lan}_{J^{\mathrm{op}}} F)(c) \otimes F'b \\
&\xRightarrow{\int \eta} \int^{bc} P_1^R(-, c, b) \otimes (\mathrm{Lan}_{J^{\mathrm{op}}} G)(c) \otimes F'b \\
&\cong \mathrm{Lan}_{J^{\mathrm{op}}}(G \star F')(-)
\end{aligned}
$$

and similarly for the right hand part. It is easily seen that $\mathrm{Lan}^L_{J^{\mathrm{op}}}$ factorises through the centre of this premonoidal structure.

The internal-hom of the left-closed premonoidal structure, $[G, -] : \overline{\mathscr{C}_1} \to \widehat{\mathscr{C}_0}$ is given by

$$
[G, H](a) \cong \int_{cd} \mathscr{V}\left(P_1^L(c, a, d), \mathscr{V}\big((\mathrm{Lan}_{J^{\mathrm{op}}} G)(d), (\mathrm{Lan}_{J^{\mathrm{op}}} H)(c)\big) \right)
$$

while the right-closed structure is similar, replacing P_1^L with P_1^R. In both cases, checking we have the required adjunction is a matter of standard coend calculus e.g:

$$
\begin{aligned}
\widehat{\mathscr{C}_0}(F, [G, H]) &\cong \int_a \mathscr{V}\left(Fa, \int_{cd} \mathscr{V}\big(P_1^L(c, a, d) \otimes (\mathrm{Lan}_{J^{\mathrm{op}}} G)(d), (\mathrm{Lan}_{J^{\mathrm{op}}} H)(c)\big) \right) \\
&\cong \int_{acd} \mathscr{V}\left(Fa \otimes P_1^L(c, a, d) \otimes (\mathrm{Lan}_{J^{\mathrm{op}}} G)(d), (\mathrm{Lan}_{J^{\mathrm{op}}} H)(c) \right) \\
&\cong \int_c \mathscr{V}\left(\int^{ab} Fa \otimes P_1^L(c, a, Jb) \otimes Gb, (\mathrm{Lan}_{J^{\mathrm{op}}} H)(c) \right) \\
&\cong \int_c \mathscr{V}\left(\mathrm{Lan}_{J^{\mathrm{op}}}(F \star G)(c), (\mathrm{Lan}_{J^{\mathrm{op}}} H)(c) \right) \\
&\cong \overline{\mathscr{C}_1}(F \otimes G, H)
\end{aligned}
$$

Suppose now that $\mathrm{Lan}_{J^{\mathrm{op}}}^L$ is a closed effectful category. Then it follows that $\widehat{\mathscr{C}_0}$ is a closed monoidal category because:

$$
\begin{aligned}
\widehat{\mathscr{C}_0}\left(-,[G,\mathrm{Lan}_{J^{\mathrm{op}}}^L(=)]\right) &\cong \overline{\mathscr{C}_1}\left(\mathrm{Lan}_{J^{\mathrm{op}}}^L(-)\boxtimes G,\mathrm{Lan}_{J^{\mathrm{op}}}^L(=)\right)\\
&= \overline{\mathscr{C}_1}\left(\mathrm{Lan}_{J^{\mathrm{op}}}^L(-\otimes G),\mathrm{Lan}_{J^{\mathrm{op}}}^L(=)\right)\\
&\cong \widehat{\mathscr{C}_0}\left(-\otimes G,J^{\mathrm{op}*}(\mathrm{Lan}_{J^{\mathrm{op}}}^L(=))\right)\\
&\cong \widehat{\mathscr{C}_0}\left(-\otimes G,=\right)
\end{aligned}
$$

where $J^{\mathrm{op}*}$ is the right adjoint to $\mathrm{Lan}_{J^{\mathrm{op}}}^L$, both of which are ioo. Therefore $\mathscr{C}_0$ is a promonoidal category.

The left $\mathscr{C}_0$-proaction on $\mathscr{C}_1$ is given by $P_1^L(-,a,b) := y_0^L(a)\boxtimes y_1^L(b) = \boxtimes i_1(y_0^L(a),y_1^L(b))$ and similarly for the right. These extend the canonical proaction because:

$$
\begin{aligned}
P_1^L(-,a,Jb) = \boxtimes i_1(y_0^L(a),y_1^L(Jb)) &= \boxtimes i_1(y_0^L(a),\mathrm{Lan}_{J^{\mathrm{op}}}^L y_0^L(b)) = \boxtimes i_1(1\otimes_{\mathscr{V}}\mathrm{Lan}_{J^{\mathrm{op}}}^L)(y_0^L(a),y_0^L(b))\\
&= \mathrm{Lan}_{J^{\mathrm{op}}}^L \otimes(y_0^L(a),y_0^L(b)) = \mathrm{Lan}_{J^{\mathrm{op}}}^L P_0(-,a,b)
\end{aligned}
$$

where we have written the monoidal operation $\otimes$ on $\widehat{\mathscr{C}_0}$ and the premonoidal operation $\boxtimes$ on $\overline{\mathscr{C}_1}$ with prefix notation. $\square$

A.14   Proof of Theorem 9

Proof. J has commutative left and right actions by the monoidal $\mathscr{V}^2$-category $1_{Z\mathscr{C}} : Z\mathscr{C} \to Z\mathscr{C}$. Consider the $\mathscr{V}^2$-category $\mathsf{Tamb}(J)$ of Tambara modules on J [35, 15], whose objects are the $\mathscr{V}^2$-endoprofunctors $P : J \nrightarrow J$ equipped with left and right strengths over the action by $1_{Z\mathscr{C}}$. The morphisms are the bistrong $\mathscr{V}^2$-natural transformations. We can use Proposition 4 to unpack $\mathsf{Tamb}(J)$ into two $\mathscr{V}$-categories and an identity on objects functor, $\mathsf{Tamb}(J)_0 \to \mathsf{Tamb}(J)_1$. The objects of $\mathsf{Tamb}(J)_0$ and $\mathsf{Tamb}(J)_1$ are the bistrong endoprofunctors $P : J \nrightarrow J$ which are equivalently triples $(P_0 : Z\mathscr{C} \nrightarrow Z\mathscr{C}, P_1 : \mathscr{C} \nrightarrow \mathscr{C}, \eta : P_0 \Rightarrow P_1(J^{\mathrm{op}}\otimes J))$. $\mathsf{Tamb}(J)_0$ has arrows $\phi : P \Rightarrow Q$ given by pairs $(\phi_0 : P_0 \Rightarrow Q_0, \phi_1 : P_1 \Rightarrow Q_1)$ while $\mathsf{Tamb}(J)_1$ has only the ϕ_1 as arrows.

It is known that the category of Tambara modules is equivalent to the presheaf category of the category of optics [35, 15], which in this particular case implies $[\mathsf{Optic}(J)^{\mathrm{op}},\mathscr{V}^2] \cong \mathsf{Tamb}(J)$. The $\mathscr{V}^2$-category $\mathsf{Optic}(J)$ has objects given by pairs $\mathbf{a} = (a,a')$ of $\mathsf{Optic}(J)$ and homs given by

$$
\mathsf{Optic}(J)(\mathbf{a},\mathbf{b}) = \int^{xy\in 1_{Z\mathscr{C}}} J(a,x\otimes b\otimes y)\otimes J(x\otimes b'\otimes y,a')
$$

where $J(-,-) := Z\mathscr{C}(-,-) \to \mathscr{C}(-,-)$ is the hom of J as a $\mathscr{V}^2$-category and the coend is taken in this fully enriched setting. By Proposition 5 this coend is given by the following arrow.

$$
\int^{xy} Z\mathscr{C}(a,x\otimes b\otimes y)\otimes Z\mathscr{C}(x\otimes b'\otimes y,a') \to \int^{xy\in Z\mathscr{C}} \mathscr{C}(a,x\otimes b\otimes y)\otimes \mathscr{C}(x\otimes b'\otimes y,a')
$$

As a result, the identity on objects functor equivalent to $\mathsf{Optic}(J)$ is given by $\mathsf{Optic}(Z\mathscr{C}) \to \mathsf{Optic}_{Z\mathscr{C}}(\mathscr{C})$ as expected.

Now, since $\mathsf{Tamb}(J)$ has a closed monoidal structure given by composition of the profunctors, there is an induced promonoidal structure on $\mathsf{Optic}(J)$. To arrive at the explicit expressions claimed in the

Theorem, take objects **a** and **b** of $\mathsf{Optic}(J)$ and consider the tensor (i.e. composition as profunctors) of the associated representable presheaves.

$$(y_{\mathbf{a}} \otimes y_{\mathbf{b}})(-) \cong \int^{wxyz \in 1_{Z\mathscr{C}}} J(-, w \otimes a \otimes x) \otimes J(w \otimes a' \otimes x, y \otimes b \otimes z) \otimes J(y \otimes b' \otimes z, -)$$

This can be unpacked by Proposition 5 to give the result.

Finally note that the unit of the monoidal structure on $\mathsf{Tamb}(J)$ is $1_J : J \rightarrow J$, which is $(1_{Z\mathscr{C}}, 1_{\mathscr{C}}, \eta : 1_{Z\mathscr{C}} \Rightarrow y^J y_J)$. $\qquad\square$

A.15   Proof of Theorem 10

Proof. The free tight cocompletion of $\mathsf{Optic}(J)$ is given by $[\mathsf{Optic}(Z\mathscr{C})^{\mathrm{op}}, \mathscr{V}] \to \overline{\mathsf{Optic}_{Z\mathscr{C}}(\mathscr{C})}$. We will show that this is a closed effectful category and then by Theorem 8 we will be done.

Start by considering the effectful category $J^{\mathrm{op}} \otimes J : Z\mathscr{C}^{\mathrm{op}} \otimes Z\mathscr{C} \to \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C}$. The free tight cocompletion of this category is $\mathsf{Lan}^L_{J^{\mathrm{op}} \otimes J} : \mathsf{Prof}(Z\mathscr{C}) \to \overline{\mathsf{Prof}(\mathscr{C})}$ which is closed effectful. The domain is the duoidal category $\mathsf{Prof}(Z\mathscr{C})$ of endoprofunctors on $Z\mathscr{C}$ and it has a closed monoidal structure given by Day convolution over the monoidal structure of $Z\mathscr{C}$:

$$P * Q := \int^{aa'bb'} Z\mathscr{C}(-, a \otimes a') \otimes P(a,b) \otimes Q(a',b') \otimes Z\mathscr{C}(b \otimes b', -) \tag{8}$$

The premonoidal structure on $\overline{\mathsf{Prof}(\mathscr{C})}$ is given on objects by (8), and on homs it acts analogously to (3). Explicitly, given a $\eta : P \Rightarrow P'$ in $\overline{\mathsf{Prof}(\mathscr{C})}$ (that is, a $\eta : \mathsf{Lan}_{J^{\mathrm{op}} \otimes J} P \Rightarrow \mathsf{Lan}_{J^{\mathrm{op}} \otimes J} P'$) the left side of the premonoidal structure is given by:

$$\mathsf{Lan}_{J^{\mathrm{op}} \otimes J}(P * Q) \cong \int^{aa'bb'} \mathscr{C}(-, J(a \otimes a')) \otimes P(a,b) \otimes Q(a',b') \otimes \mathscr{C}(J(b \otimes b'), -)$$

$$\cong \int^{a'b' \in Z\mathscr{C}, cd \in \mathscr{C}} \mathscr{C}(-, c \rtimes a') \otimes (\mathsf{Lan}_{J^{\mathrm{op}} \otimes J} P)(c,d) \otimes Q(a',b') \otimes \mathscr{C}(d \rtimes b', -)$$

$$\xRightarrow{\int \eta} \int^{a'b' \in Z\mathscr{C}, cd \in \mathscr{C}} \mathscr{C}(-, c \rtimes a') \otimes (\mathsf{Lan}_{J^{\mathrm{op}} \otimes J} P')(c,d) \otimes Q(a',b') \otimes \mathscr{C}(d \rtimes b', -)$$

$$\cong \mathsf{Lan}_{J^{\mathrm{op}} \otimes J}(P' * Q)$$

Since $\mathsf{Lan}^L_{J^{\mathrm{op}} \otimes J}$ is a left adjoint, it follows that it is a closed effectful category.

There is a $\mathscr{V}^2$-category $\mathsf{Tamb}(Z\mathscr{C}) \to \overline{\mathsf{Tamb}(\mathscr{C})}$ with objects given by the Tambara modules on $Z\mathscr{C}$. The homs of $\mathsf{Tamb}(Z\mathscr{C})$ are the bistrong natural transformations while the homs of $\overline{\mathsf{Tamb}(\mathscr{C})}$ are the bistrong natural transformations between the left Kan extensions along $J^{\mathrm{op}} \otimes J$ of the Tambara modules. This $\mathscr{V}^2$-category inherits a closed effectful structure from $\mathsf{Lan}^L_{J^{\mathrm{op}} \otimes J}$ given by a certain quotient of (8) which acts to normalise the duoidal structure on $\mathsf{Prof}(Z\mathscr{C})$ [24, 22].

Finally note that the presheaf category of optics is equivalent to the category of Tambara modules, $\widehat{\mathsf{Optic}(Z\mathscr{C})}^{\mathrm{op}} \cong \mathsf{Tamb}(Z\mathscr{C})$ [15], and we can finally check that we also have $\overline{\mathsf{Optic}_{Z\mathscr{C}}(\mathscr{C})} \cong \overline{\mathsf{Tamb}(\mathscr{C})}$. $\qquad\square$

B   Background on Profunctors and Promonads

B.1   Profunctors

Here we assemble a very brief summary of the main definitions of profunctors and promonads, a more comprehensive discussion can be found in e.g. [31, 34].

Definition 16 (Profunctor)**.** A profunctor $P : \mathscr{C} \nrightarrow \mathscr{D}$ is a functor $\mathscr{D}^{\mathrm{op}} \otimes \mathscr{C} \to \mathscr{V}$.

There are a few ways to conceptualise the notion of a profunctor. If we think of a functor as a categorification of a function, then a profunctor can be thought to be like a categorification of a binary relation - indeed every relation $A \sim B$ between sets is a function $A \times B \to 2$. Similarly to how every function is a relation, profunctors are strict generalisations of functors, for every functor $F : \mathscr{C} \to \mathscr{D}$ induces two profunctors given by its contravariant and covariant Yoneda embeddings: $\mathscr{D}(-, F=) : \mathscr{C} \nrightarrow \mathscr{D}$ and $\mathscr{D}(F-, =) : \mathscr{D} \nrightarrow \mathscr{C}$.

Definition 17 (Extranatural Transformation, Coend)**.** Given a profunctor $P : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \to \mathscr{V}$, a family of arrows $w_c : P(c, c) \to d$ in $\mathscr{V}$ is extranatural if the left-hand diagram commutes for all c and c'.

$$
\begin{array}{ccc}
\mathscr{C}(c, c') \otimes P(c', c) & \xrightarrow{\ r\ } & P(c', c') \\
{\scriptstyle l}\downarrow & & \downarrow{\scriptstyle w_{c'}} \\
P(c, c) & \xrightarrow[\ w_c\ ]{} & d
\end{array}
\qquad\qquad
\begin{array}{ccc}
P(c', c) & \xrightarrow{\ P(c', f)\ } & P(c', c') \\
{\scriptstyle P(f, c)}\downarrow & & \downarrow{\scriptstyle w_{c'}} \\
P(c, c) & \xrightarrow[\ w_c\ ]{} & d
\end{array}
$$

Here the arrows l and r are the left and right actions of the hom on P, given by transporting the arrows defining P as an enriched functor along the adjunction due to the closed monoidal structure of $\mathscr{V}$. In the case where $\mathscr{V} = \mathsf{Set}$, the extranaturality condition reduces to the usual "cowedge" diagram on the right above. The coend of P is a universal extranatural transformation $\mathrm{copr}_c : P(c, c) \to \int^c P(c, c)$: this is the extranatural transformation such that all other extranatural transformations factorise uniquely through it.

The composition of profunctors is given by coend:

$$
Q \circ P = \int^d Q(-, d) \otimes P(d, -)
$$

where the identity profunctor is given by the hom-profunctor $\mathscr{C}(-, =) : \mathscr{C} \nrightarrow \mathscr{C}$ because of the following instances of the "ninja" Yoneda lemma,

$$
F(-) \cong \int^c \mathscr{C}(-, c) \otimes F(c), \qquad G(-) \cong \int^c G(c) \otimes \mathscr{C}(c, -)
$$

Categories, profunctors and natural transformations assemble into a monoidal bicategory $\mathscr{V}\text{-}\mathsf{Prof}$.

It will be useful to also outline another way to view profunctors and their composition. The Yoneda lemma gives us an embedding $y : \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$ which exhibits the presheaf category $\widehat{\mathscr{C}} := [\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$ as the free cocompletion of $\mathscr{C}$. In particular, for any cocomplete category $\mathscr{D}$ and functor $F : \mathscr{C} \to \mathscr{D}$ there is a unique (up to natural isomorphism) cocontinuous functor $F' : \widehat{\mathscr{C}} \to \mathscr{D}$ such that $F'y \cong F$ naturally. We can construct such a functor by taking the left Kan extension $F' \cong \mathrm{Lan}_y F$.

Now, using the closed monoidal structure of $\mathscr{V}\text{-}\mathsf{Cat}$ we can see that every profunctor $P : \mathscr{C} \nrightarrow \mathscr{D}$ is equivalently a functor $P' : \mathscr{C} \to \widehat{\mathscr{D}}$. We can then lift P to a functor $\widehat{P} := \mathrm{Lan}_y P' : \widehat{\mathscr{C}} \to \widehat{\mathscr{D}}$ which is cocontinuous. That is to say, P is equivalent to a cocontinuous functor between presheaf categories. The composition of profunctors can now be shown to be composition of the corresponding cocontinuous functors. In fact, there is an equivalence of bicategories $\mathscr{V}\text{-}\mathsf{Prof} \cong \mathscr{V}\text{-}\mathsf{Cocont}$ between the bicategory of profunctors and the 2-category of cocontinuous functors between free cocompletions.

B.2   Promonads

While a monad is a monoid in the category of endofunctors, a promonad is a monoid in the category of endoprofunctors. Unpacking this yields the following definition:

Definition 18 (Promonad). A promonad is a triple (T, μ, η) of a profunctor $T : \mathscr{C} \nrightarrow \mathscr{C}$, a natural transformation $\mu : T^2 = T \circ T \to T$ and a natural transformation $\eta : 1 \to T$ such that the following diagrams commute:

$$
\begin{array}{ccc}
T^3 & \xrightarrow{T\mu} & T^2 \\
{\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\
T^2 & \xrightarrow{\mu} & T
\end{array}
\qquad
\begin{array}{ccc}
T & \xrightarrow{T\eta} & T^2 \\
{\scriptstyle \eta T}\downarrow & \big\Vert & \downarrow{\scriptstyle \mu} \\
T^2 & \xrightarrow{\mu} & T
\end{array}
$$

Example 1. Let $T : \mathscr{C} \to \mathscr{C}$ be a monad. Then its contravariant Yoneda embedding $\mathscr{C}(-, T=) : \mathscr{C} \nrightarrow \mathscr{C}$ yields a promonad on $\mathscr{C}$.

Conceptually it may be useful to think of a promonad T as a generalised hom-functor: for each pair of objects A, B there is a set $T(A, B)$ of "arrows" from A to B together with a composition rule given by μ and a unit rule given by η.

Any promonad can also be viewed as a cocontinuous monad because of the equivalence Prof $\cong$ Cocont. Given a promonad T, we can view T instead as a cocontinuous functor $\widehat{T} : \widehat{\mathscr{C}} \to \widehat{\mathscr{C}}$. The promonad laws of T can then be shown to make $\widehat{T}$ into a monad. Conversely, any cocontinuous monad $\widehat{T} : \widehat{\mathscr{C}} \to \widehat{\mathscr{C}}$ is equivalently a promonad $T : \mathscr{C} \nrightarrow \mathscr{C}$.

The bicategory $\mathscr{V}$-Prof permits the Kleisli construction for monads [46] so that we can assign any promonad a Kleisli category.

Definition 19 (Kleisli category). Let $T : \mathscr{C} \nrightarrow \mathscr{C}$ be a promonad. The Kleisli category Kl_T has the same objects as $\mathscr{C}$ and hom-objects $\mathsf{Kl}_T(a, b) = T(a, b)$. Composition is induced by the multiplication μ of the promonad, and units by η.

The unit η of the promonad T also induces an identity on objects functor $F : \mathscr{C} \to \mathsf{Kl}_T$. Conversely, given any identity on objects functor $J : \mathscr{C} \to \mathscr{D}$ one can construct a promonad $\mathscr{D}(J-, J=) : \mathscr{C} \nrightarrow \mathscr{C}$. These constructions are mutually inverse so that $\mathsf{Kl}_T(F-, F=) = T(-, =)$.

Example 2. For simplicity take $\mathscr{V} = \mathsf{Set}$. Let $T : \mathscr{C} \to \mathscr{C}$ be a monad and write $T' = \mathscr{C}(-, T=) : \mathscr{C} \nrightarrow \mathscr{C}$ for the induced promonad. The Kleisli category of T' has the same objects as $\mathscr{C}$ while an arrow $a \to b$ is an element of $\mathscr{C}(a, Tb)$, that is an arrow $a \to Tb$. We can see that $\mathsf{Kl}_{T'}$ coincides with the standard Kleisli category Kl_T of the monad T.

Now when $\mathscr{C}$ is promonoidal one can define a prostrength for a promonad on $\mathscr{C}$ so as to ensure that the promonad behaves compatibly with the promonoidal structure.

Definition 20 (Prostrong Promonad). Suppose $\mathscr{C}$ is a promonoidal category and suppose (T, μ, η) is a promonad on $\mathscr{C}$. A left prostrength for T is a natural transformation
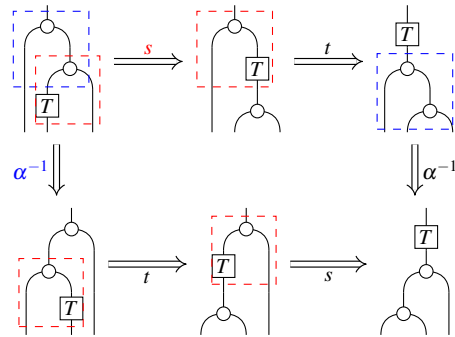

such that the following diagrams commute (ignoring interchange isomorphisms in $\mathscr{V}$-Prof).
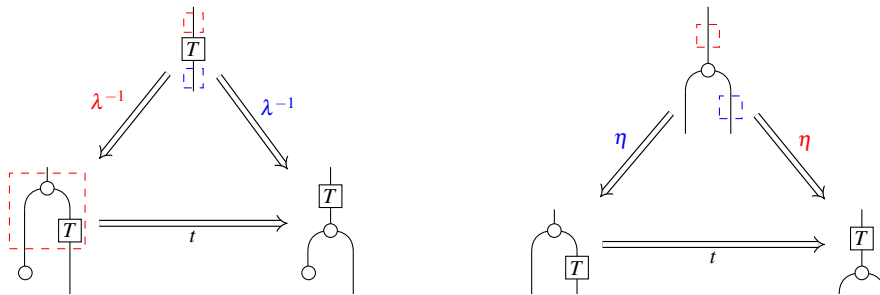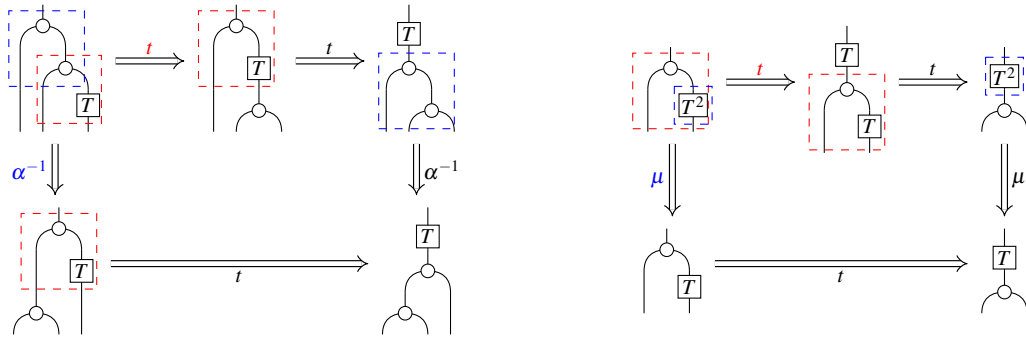
A right prostrength *s* is defined analogously and we say that a promonad is prostrong if it is equipped with left and right prostrengths such that the two evident maps agree:

When $\mathscr{C}$ is monoidal (so that the promonoidal structure is representable) we recover the notion of a strong promonad. These are also known as "arrows" because they axiomatise such objects in functional programming [3, 27].