Fox-cartesian structure for Gray-monoidal double categories

Edward Morehouse*

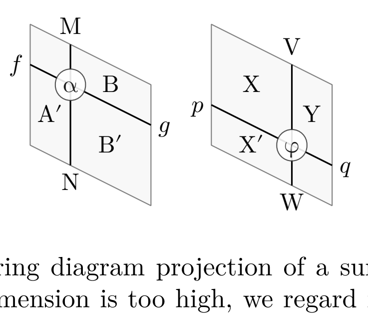
Tallinn University of Technology

This talk presents some results from the article, *Cartesian Gray-Monoidal Double Categories* [Mor23]. There, the notion of a *locally cubical Gray category* is proposed, and it is shown that double categories with a hierarchy of their morphisms, as well as classical (locally globular) Gray categories, are instances of this construction. A one-object locally cubical Gray category is a *Gray-monoidal double category* [Böh19], which can be endowed with *braided*, *sylleptic*, and *symmetric* structure, as in the globular case [KV94; BN96; DS97; Cra98]. Equipping a symmetric Gray-monoidal double category with Fox-cartesian structure [Fox76] requires compatible and coherent duplication and deletion. This necessitates the introduction of *doubly-lax functors*, along with multiple *duplicator* transformations, and *coassociator* and *cocommutor* modifications for these. Intuitively, the reason for this is that in the Gray-monoidal setting we cannot do two things at once, only one at a time, so we must impose and maintain an order on the higher-dimensional cells. An algebraic presentation of the resulting theory is rather complex due to the bureaucracy of linearizing higher-dimensional boundary constraints. Fortunately, it has a relatively simple and compelling representation in the graphical calculus of *surface diagrams*, which we present.

1 Gray-monoidal double categories

In the globular setting, Gray constructed a tensor product functor for 2-categories, showed it to be left adjoint to the internal hom, and used it to define Gray-monoidal structure on a 2-category [Gra74]. In the cubical setting, Böhm characterized the corresponding functor for (strict) double categories, and used it to define Gray-monoidal structure on a double category [Böh19]. In [Mor23] I characterize Gray-monoidal structure on a (preunitary weak) double category as a one-object locally cubical Gray category, a 3-dimensional categorical structure that is cubical in two dimensions and globular in the third, and which generalizes a classical, locally globular, Gray category.

Intuitively, Gray-monoidal structure is *dimension-summing* rather than *dimension-sup'ing*, in the sense that the tensor product of an m-cell with an n-cell is an $(m+n)$-cell rather than an $(m \vee n)$-cell, when $m+n$ does not exceed the number of dimensions available. The structure of Gray-monoidal double categories is intuitively conveyed in the graphical language of surface diagrams. For example, the surface diagram below on the left, with projection string diagram on the right, depicts a configuration of four 2-cells (or "squares") that can be written as two equivalent expressions involving only binary compositions. The equivalence of these expressions, known as *middle-four exchange*, can be regarded as a diagrammatic associative law. The unlabeled squares, depicted as crossings, are Gray *interchangers* of their boundary 1-cells.



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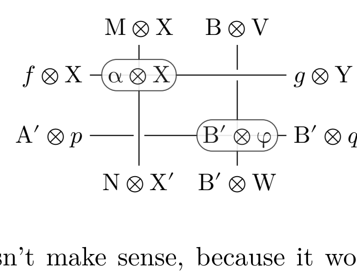

When the string diagram projection of a surface diagram doesn't make sense, because it would contain a cell whose dimension is too high, we regard it as a relation identifying the admissible boundary-preserving

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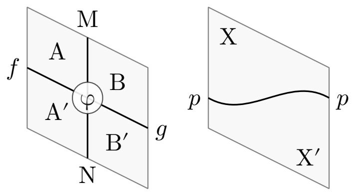
perturbations of the surface diagram. For example, the surface diagram on the left represents the relation between expressions represented by the projection string diagrams on the right, where we write “≅” rather than “=” to indicate that the boundaries must be unified by conjugating with unitors.



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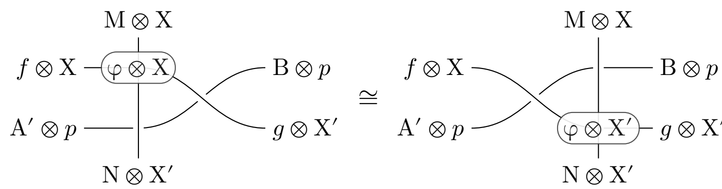
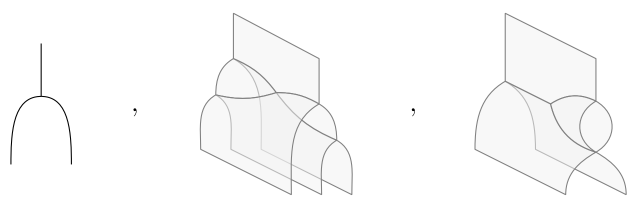


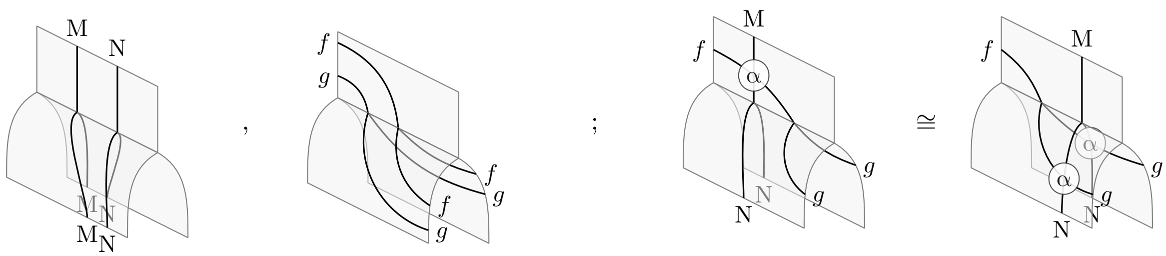
Once we have Gray-monoidal structure for a double category we can add a *braiding* transformation and *syllepsis* modification, as in the case of 2-categories. Their object-components are depicted on the left. A syllepsis is a *symmetry* just in case it is the unit of an adjoint equivalence, as depicted on the right.

2 Fox-cartesian structure

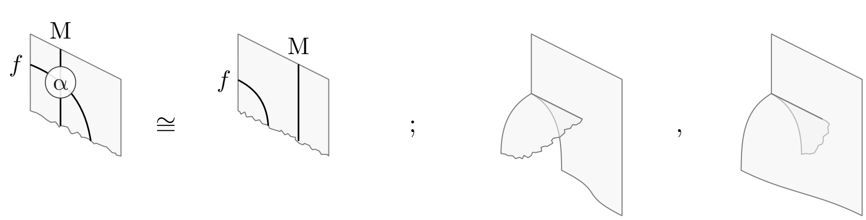
A duplication structure for a symmetric Gray-monoidal double category consists of oplax *duplicator* transformations, together with *coassociator* and *cocommutor* modifications, with the object-components shown.



We need multiple duplicator transformations because we cannot tensor two nonzero-dimensional cells, so we must choose an order for these cells in each dimension. The interchangers and braidings permute this order, and the coassociators and cocommutors must be coherent with respect to it. Duplicator transformations must mediate between *doubly-lax functors* in order to be compatible with 1-cell composition, as shown on the left. Their naturality for squares is shown on the right.



For *Fox-cartesian* structure we also require a *deletor* transformation whose naturality for squares is shown on the left, and coherent *counitor* modifications with object-components shown on the right.



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