Subsumptions of Algebraic Rewrite Rules

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What does it mean for an algebraic rewrite rule to subsume another rule (that may then be called a subrule)? We view subsumptions as rule morphisms such that the simultaneous application of a rule and a subrule (i.e. the application of a morphism) yields the same result as a single application of the rule. Simultaneous applications of categories of rules are obtained by Global Coherent Transformations and illustrated on graphs in the DPO approach. Other approaches are possible since these transformations are formulated in an abstract Rewriting Environment, and such environments exist for DPO, SqPO, PBPO rules.

1 Introduction

In Global Transformations [17] rules in the form of pairs (L,R) of graphs (or objects in a category $\mathscr{C}$) are applied simultaneously to an input graph (as in L-systems [10] and cellular automata [9]). Such rules are related by pairs of $\mathscr{C}$-morphisms. These morphisms come from representing possible overlaps of rules as subrules whose applications are induced by the overlapping applications of rules, therefore establishing a link between these. By computing a colimit of a diagram involving the morphisms between occurrences of right-hand sides, Global Transformations offer the possibility to merge items (vertices or edges) in these occurrences of right-hand sides.

This form of rules has the advantage of simplicity, first because rule morphisms are those of the product category $\mathscr{C} \times \mathscr{C}$, and second because the input object is completely removed. Indeed, when all occurrences of L have been found in the input graph G, the output graph H is produced solely from the corresponding occurrences of R, thus effectively removing G. In particular, if no L has any match in G then H is the empty graph. If G is, say, a relational database, this may be inconvenient.

More standard approaches to rewriting use rules for *replacing* matched parts of the input object by new parts. These substitutions are performed by first removing the matched part and then adding the new part, which is performed by a pushout. But since there is no general algebraic way of removing parts of a $\mathscr{C}$-object, several approaches have been devised, from DPO [7] to PBPO [4] rules. These rules always have an interface K with a pair of $\mathscr{C}$-morphisms from K to L and R, but can be more complicated. Hence the necessity of a general notion of morphism between rules.

In Section 3 an intuitive analysis of rule subsumptions on a simple example with DPO-rules leads to morphisms between DPO direct transformations. The intuition is made clear in Section 4 where Global Coherent Transformations are defined and illustrated on the running example with DPO rules. This definition is carried out in a *Rewriting Environment* that provides the relevant categories of rules and of direct transformations (and functors between them). This derives from the Parallel Coherent Transformations defined in [2] (only for a variant of DPO rules), where sets or rules can be applied simultaneously on an input object. One important problem is that overlapping rules may conflict if one rule deletes an item that another rule preserves (this cannot happen with Global Transformations since they preserve nothing). Since only non conflicting (so called *coherent*) matchings can be applied simultaneously, the notion of Parallel Coherence from [2] is adapted in order to embrace rule morphisms.

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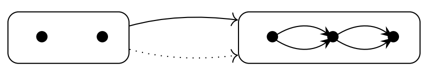
Section 5 is devoted to the analysis of Rewriting Environments, and yields natural definitions of environments for the SqPO and PBPO approaches. Future work and open questions are found in Section 6.

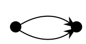
2 Notations

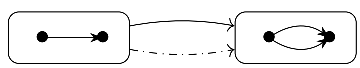
Embeddings are injective functors (as in [11]), all other notions are compatible with [16]. We also use *meets* and *sums* of functors, see [13].

For any category $\mathscr{C}$, we write $G \in \mathscr{C}$ to indicate that G is a $\mathscr{C}$-object, and $|\mathscr{C}|$ is the discrete category on $\mathscr{C}$-objects. Then G also denotes the functor from the terminal category $\mathbf{1}$ to $|\mathscr{C}|$ that maps the object of $\mathbf{1}$ to G. The *slice* category $\mathscr{C} \setminus G$ has as objects $\mathscr{C}$-morphisms of codomain G, and as morphisms $h : f \to g$ $\mathscr{C}$-morphisms such that $g \circ h = f$. The *coslice* category $G \setminus \mathscr{C}$ has as objects $\mathscr{C}$-morphisms of domain G, and as morphisms $h : f \to g$ $\mathscr{C}$-morphisms such that $h \circ f = g$.

We will use the standard notion of graphs with multiple directed edges. The initial graph is denoted $\varnothing$. In the running example we will use graphs with 2 to 3 vertices and 0 to 4 edges denoted directly by their drawings, as in and . In order to avoid naming vertices, they will always be depicted from left to right, and we will use at most two monomorphisms from one graph to another: one (depicted as a plain arrow) that maps the leftmost (resp. rightmost) vertex of the domain graph to the leftmost (resp. rightmost) vertex of the codomain graph, and one (dotted arrow) that swaps these vertices. For example we consider only two possible morphisms:



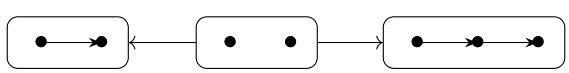
The two morphisms from to  will be distinguished similarly:



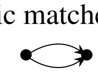
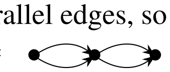
3 Subrules in DPO Graph Transformations

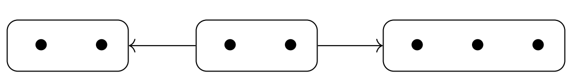
The notion of a rule ρ being a subrule of a rule ρ', or more generally of a morphism $\sigma : \rho \to \rho'$, covers the idea that ρ represents a part (specified by σ) of what ρ' achieves, and therefore that any application of ρ' entails and subsumes a particular application (obtained through σ) of ρ. In [17] the morphisms σ are pairs of unrelated graph morphisms. But in the DPO approach the left and right-hand sides of a rule are related by morphisms from an interface graph K, hence the graph morphisms in σ should also be related.

Example 3.1. In the running example we transform every directed edge in a graph into a pair of consecutive edges. This can be expressed as the following rule



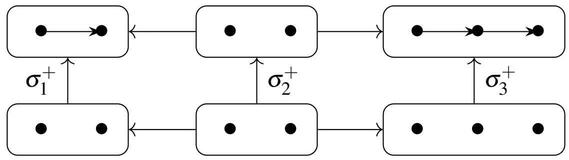
(ρ')

We do not wish to transform loops in this way, hence we adopt the DPO approach restricted to monic matches. We also wish to create only one middle vertex for parallel edges, so that the input graph $G =$  in our running example shall be transformed into $H =$ . In order to merge the two vertices created by the two simultaneous applications of ρ' on G we need to link them through the application of a common subrule on their overlap. Consider the rule



(ρ)

The right hand side expresses the fact that the middle vertex is created depending on the overlap $\bullet \quad \bullet$ and not on the edges of G. Thus we need to link the middle vertices from ρ and ρ' right-hand sides through a morphism $\sigma^+ : \rho \to \rho'$, given as three $\mathscr{C}$-morphisms:



(σ^+)

The two square diagrams commute, and we easily understand that this is necessary for ρ to be a subrule of ρ'. But commutation would also hold if the interface graph of ρ were $\varnothing$, and then ρ would remove the overlap $\bullet \quad \bullet$. This would conflict with ρ' that preserves the overlap. We need the two rules to behave similarly on the overlap, which means that the interface of the subrule ρ is determined by the way the interface of ρ' intersects the overlap. This can be expressed by stating that the left square should be a pullback.

Definition 3.2 (categories $\mathscr{R}_{\mathrm{DPO}}, \mathscr{R}_{\mathrm{DPOm}}$)**.** Let **sp** be the category generated by the graph

$$\mathrm{L} \xleftarrow{\mathrm{l}} \mathrm{K} \xrightarrow{\mathrm{r}} \mathrm{R}$$

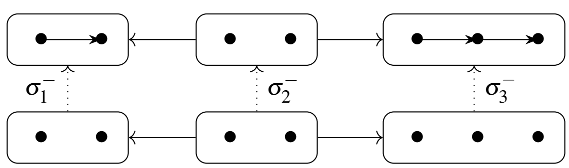
For any category $\mathscr{C}$, let $\mathscr{R}_{\mathrm{DPO}}$ be the category whose objects are the functors[1] $\rho : \mathbf{sp} \to \mathscr{C}$ such that $\rho\mathrm{l}$ is monic, and morphisms $\sigma : \rho \to \rho'$ are triples $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ of $\mathscr{C}$-morphisms such that

$$\begin{array}{ccccc} \rho\mathrm{L} & \xleftarrow{\rho\mathrm{l}} & \rho\mathrm{K} & \xrightarrow{\rho\mathrm{r}} & \rho\mathrm{R} \\ {\scriptstyle\sigma_1}\downarrow & \lrcorner & \downarrow{\scriptstyle\sigma_2} & & \downarrow{\scriptstyle\sigma_3} \\ \rho'\mathrm{L} & \xleftarrow[\rho'\mathrm{l}]{} & \rho'\mathrm{K} & \xrightarrow[\rho'\mathrm{r}]{} & \rho'\mathrm{R} \end{array}$$

commutes in $\mathscr{C}$ and the left square is a pullback. The obvious composition $\sigma' \circ \sigma$ is given by $(\sigma_1' \circ \sigma_1, \sigma_2' \circ \sigma_2, \sigma_3' \circ \sigma_3)$, and the obvious identity is $1_\rho = (1_{\rho\mathrm{L}}, 1_{\rho\mathrm{K}}, 1_{\rho\mathrm{R}})$.

Let $\mathscr{R}_{\mathrm{DPOm}}$ be the subcategory of $\mathscr{R}_{\mathrm{DPO}}$ with all rules and morphisms σ such that σ_1 and σ_2 are monics.

Example 3.3. We consider two morphisms of rules, σ^+ above and $\sigma^- : \rho \to \rho'$ that swaps the left and right vertices:



(σ^-)

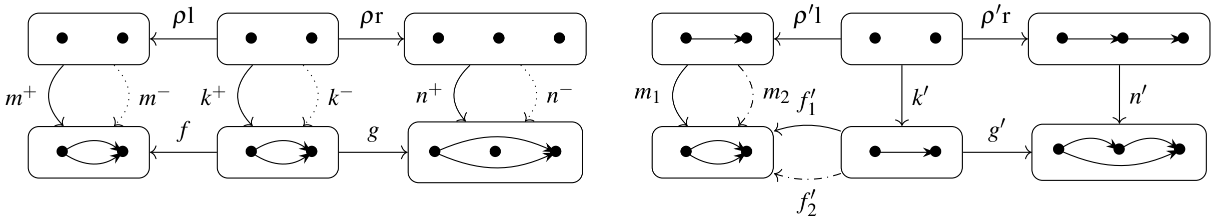
There are two obvious matchings m_1 and m_2 of ρ' in G, and they induce two matchings of ρ in G, say $m^+ = m_1 \circ \sigma_1^+ = m_2 \circ \sigma_1^+$ and $m^- = m_1 \circ \sigma_1^- = m_2 \circ \sigma_1^-$.

The existence of DPO-transformations that correspond to these matchings requires the existence of pushout complements and is subject to the gluing condition (see [8, 6]). We say that an item (edge or vertex) of G is *marked for removal* by $m : \rho\mathrm{L} \to G$ if it has a preimage by the matching m that has none by $\rho\mathrm{l}$ (those are the items that have no preimage in the pushout complement, see [3]). Then the *gluing condition* states that (1) items marked for removal have only one preimage by the matching m, and (2) if a vertex adjacent to an edge is marked for removal, then so is this edge. We first see that this condition is inherited (backward) along the morphisms of $\mathscr{R}_{\mathrm{DPOm}}$.

[1] Such functors are span diagrams in $\mathscr{C}$, the point here is to give fixed names to their objects and morphisms.

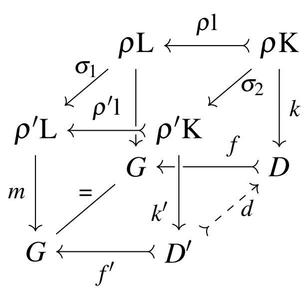
Proposition 3.4. *If $\mathscr{C}$ is the category of graphs, $\sigma : \rho \to \rho'$ is a morphism in $\mathscr{R}_{\mathrm{DPO}}$ such that σ_1 is monic and $m : \rho'\mathrm{L} \to G$ is a matching that satisfies the gluing condition then so does the matching $m \circ \sigma_1 : \rho\mathrm{L} \to G$.*

Example 3.5. We see that m_1 and m_2 satisfy the gluing condition, hence they have a pushout complement by $\rho'\mathrm{l}$ and so do m^+ and m^- by $\rho\mathrm{l}$. We therefore get two DPO transformations of G by ρ (below left), one with (m^+, k^+, n^+, f, g) and the other with (m^-, k^-, n^-, f, g), and two DPO transformations of G by ρ' (below right), one with (m_1, k', n', f'_1, g') and the other with (m_2, k', n', f'_2, g').



The following result reveals the relationship induced by morphisms $\sigma : \rho \to \rho'$ on the corresponding direct DPO transformations.

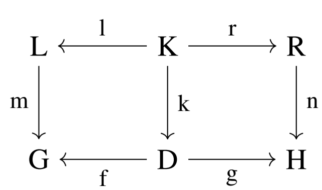
Proposition 3.6. *If $\mathscr{C}$ is the category of graphs, $\sigma : \rho \to \rho'$ is a morphism in $\mathscr{R}_{\mathrm{DPO}}$, $m : \rho'\mathrm{L} \to G$ and $m \circ \sigma_1 : \rho\mathrm{L} \to G$ have pushout complements as below, then there is a unique graph morphism d such that*



commutes.

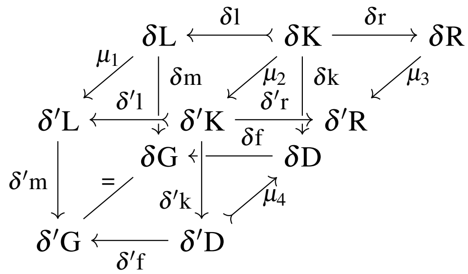
The existence of d means that all items marked for removal by $m \circ \sigma_1$, i.e., removed by the subrule ρ, are also removed by ρ'. In Example 3.5 we have $f = 1_G$, hence with $m = m_i$ we get $d = f'_i$. We also see that there are no morphisms between the results of the transformations of G by ρ and ρ', in either direction. This is due to the fact that subrules remove less, but also add less. Hence contrary to Global Transformations, subsumptions of rules cannot be deduced from the properties of the transformation functions (from $\mathscr{C}$ to $\mathscr{C}$) they induce.

Definition 3.7 (categories $\mathscr{D}_{\mathrm{DPO}}$, $\mathscr{D}_{\mathrm{DPOm}}$, functors $\mathsf{R}_{\mathrm{DPO}}$, $\mathsf{R}_{\mathrm{DPOm}}$)**.** Let **dt** be the category generated by the graph



with relations $\mathrm{m} \circ \mathrm{l} = \mathrm{f} \circ \mathrm{k}$ and $\mathrm{n} \circ \mathrm{r} = \mathrm{g} \circ \mathrm{k}$. A *direct DPO-transformation* in $\mathscr{C}$ is a functor $\delta : \mathbf{dt} \to \mathscr{C}$ such that $\delta\mathrm{l}$ is monic and the two squares $(\delta\mathrm{l}, \delta\mathrm{m}, \delta\mathrm{k}, \delta\mathrm{f})$, $(\delta\mathrm{r}, \delta\mathrm{n}, \delta\mathrm{k}, \delta\mathrm{g})$ are pushouts.

Let $\mathscr{D}_{\mathrm{DPO}}$ be the category whose objects are direct DPO-transformations in a category $\mathscr{C}$ and whose morphisms $\mu : \delta \to \delta'$ are 4-tuples $(\mu_1, \mu_2, \mu_3, \mu_4)$ of $\mathscr{C}$-morphisms such that μ_4 is monic, the following diagram



commutes and the top left square is a pullback, with obvious composition and identities. Let $\mathsf{R}_{\mathrm{DPO}}$ be the obvious functor from $\mathscr{D}_{\mathrm{DPO}}$ to $\mathscr{R}_{\mathrm{DPO}}$, i.e. such that $(\mathsf{R}_{\mathrm{DPO}}\delta)\mathrm{L} = \delta\mathrm{L}$ etc. and $\mathsf{R}_{\mathrm{DPO}}\mu = (\mu_1, \mu_2, \mu_3)$. Let $\mathscr{D}_{\mathrm{DPOm}}$ be the full subcategory of $\mathscr{D}_{\mathrm{DPO}}$ whose objects are the direct transformations δ such that $\delta\mathrm{m}$ is monic, and let $\mathsf{R}_{\mathrm{DPOm}} : \mathscr{D}_{\mathrm{DPOm}} \to \mathscr{R}_{\mathrm{DPOm}}$ be the corresponding restriction of $\mathsf{R}_{\mathrm{DPO}}$.

4 Global Coherent Transformations

The result of simultaneous rewriting will be computed exclusively from data extracted from all relevant direct transformations as follows: we extract the span $D \xleftarrow{k} K \xrightarrow{r} R$ from which the result of the transformation is obtained (as its pushout), and we also keep the link $G \xleftarrow{f} D$ to the input G.

Definition 4.1 (category $\mathscr{C}_{\mathrm{pt}}$, functors In, $\mathsf{P}_{\mathrm{DPOm}}$)**.** Let $\mathbf{pt}$ be the category generated by the graph

$$\mathrm{G} \xleftarrow{\ \mathrm{f}\ } \mathrm{D} \xleftarrow{\ \mathrm{k}\ } \mathrm{K} \xrightarrow{\ \mathrm{r}\ } \mathrm{R}$$

For any category $\mathscr{C}$, the category $\mathscr{C}_{\mathrm{pt}}$ of *partial transformations* has as objects functors $\tau : \mathbf{pt} \to \mathscr{C}$, and as morphisms $\nu : \tau \to \tau'$ triples (ν_1, ν_2, ν_3) such that

$$\begin{array}{ccccccc}
\tau\mathrm{G} & \xleftarrow{\tau\mathrm{f}} & \tau\mathrm{D} & \xleftarrow{\tau\mathrm{k}} & \tau\mathrm{K} & \xrightarrow{\tau\mathrm{r}} & \tau\mathrm{R} \\
\| & & \uparrow{\scriptstyle \nu_1} & & \downarrow{\scriptstyle \nu_2} & & \downarrow{\scriptstyle \nu_3} \\
\tau'\mathrm{G} & \xleftarrow[\tau'\mathrm{f}]{} & \tau'\mathrm{D} & \xleftarrow[\tau'\mathrm{k}]{} & \tau'\mathrm{K} & \xrightarrow[\tau'\mathrm{r}]{} & \tau'\mathrm{R}
\end{array}$$

commutes in $\mathscr{C}$, with obvious composition and identities.

Let $\mathsf{In} : \mathscr{C}_{\mathrm{pt}} \to |\mathscr{C}|$ be the *input functor* defined as $\mathsf{In}\tau = \tau\mathrm{G}$. Let $\mathsf{P}_{\mathrm{DPO}} : \mathscr{D}_{\mathrm{DPO}} \to \mathscr{C}_{\mathrm{pt}}$ and $\mathsf{P}_{\mathrm{DPOm}} : \mathscr{D}_{\mathrm{DPOm}} \to \mathscr{C}_{\mathrm{pt}}$ be the obvious functors.

Using inverse images of the functors $\mathsf{P}_{\mathrm{DPOm}}$ and $\mathsf{R}_{\mathrm{DPOm}}$ we can easily focus on the direct transformations of concern (and the morphisms between them), i.e., the transformations *of* a graph *by* a rule system.

Definition 4.2 (Rewriting Environments, rule systems, notations D_δ, $\pi_1\mu$ ...)**.** For any category $\mathscr{C}$, a *Rewriting Environment* for $\mathscr{C}$ consists of a category $\mathscr{D}$ of *direct transformations*, a category $\mathscr{R}$ of *rules* and two functors

$$\mathscr{R} \xleftarrow{\ \mathsf{R}\ } \mathscr{D} \xrightarrow{\ \mathsf{P}\ } \mathscr{C}_{\mathrm{pt}}$$

A *rule system* in a Rewriting Environment is a category $\mathscr{S}$ with an embedding $\mathsf{I} : \mathscr{S} \to \mathscr{R}$ (alternately, $\mathscr{S}$ is a subcategory of $\mathscr{R}$ and I is the inclusion functor).

Given a rule system and an *input* $\mathscr{C}$-object G, we build the categories $\mathscr{D}|_G$, $\mathscr{D}|_G^{\mathscr{S}}$ and functors I_G, $\mathsf{I}_{\mathscr{S}}$, $\mathsf{R}|_G^{\mathscr{S}}$ as meets of previous functors:

$$
\begin{array}{ccccccc}
\mathscr{S} & \overset{\mathsf{I}}{\rightarrowtail} & \mathscr{R} & & & & \\
\uparrow & & \uparrow \mathsf{R} & & & & \\
\mathsf{R}|_G^{\mathscr{S}} & & \mathscr{D} & \overset{\mathsf{P}}{\longrightarrow} & \mathscr{C}_{\text{pt}} & \overset{\mathsf{In}}{\longrightarrow} & |\mathscr{C}| \\
 & & \mathsf{I}_G \uparrow & & & & \uparrow G \\
\mathscr{D}|_G^{\mathscr{S}} & \underset{\mathsf{I}_{\mathscr{S}}}{\rightarrowtail} & \mathscr{D}|_G & & \longrightarrow & & \mathbf{1}
\end{array}
$$

For any $\delta \in \mathscr{D}|_G^{\mathscr{S}}$ we write D_δ for $(\mathsf{P}\mathsf{I}_G\mathsf{I}_{\mathscr{S}}\delta)\mathrm{D}$ and similarly f_δ etc. For any $\mu : \delta \to \delta'$ in $\mathscr{D}|_G^{\mathscr{S}}$ we write $\pi_1\mu$ for the first coordinate of $\mathsf{P}\mathsf{I}_G\mathsf{I}_{\mathscr{S}}\mu$ and similarly $\pi_2\mu$, $\pi_3\mu$.

Example 4.3. For $\mathscr{S}$ we take the subcategory $\rho \overset{\sigma^+}{\underset{\sigma^-}{\rightrightarrows}} \rho'$ of $\mathscr{R}_{\text{DPO}}$. To the matchings m_1 and m_2 of ρ' in G correspond two[2] transformations in $\mathscr{D}_{\text{DPOm}}$ that will be denoted δ'_1 and δ'_2 (their are depicted on the right of Example 3.5 with $f'_i = \delta'_i\mathrm{f}$ etc.). To the matchings m^+ and m^- of ρ in G correspond another two transformations denoted δ^+ and δ^- (on the left of Example 3.5). To each $i = 1,2$ correspond one morphism $\mu_i^+ : \delta^+ \to \delta'_i$ such that $\mathsf{R}_{\text{DPOm}}\mu_i^+ = \sigma^+$ and one morphism $\mu_i^- : \delta^- \to \delta'_i$ such that $\mathsf{R}_{\text{DPOm}}\mu_i^- = \sigma^-$.

Thus $\mathscr{D}_{\text{DPOm}}|_G^{\mathscr{S}}$ is the subcategory
$$
\begin{array}{ccccc}
 & & \delta'_1 & & \\
 & \mu_1^+ \nearrow & & \nwarrow \mu_1^- & \\
\delta^+ & & & & \delta^- \\
 & \mu_2^+ \searrow & & \swarrow \mu_2^- & \\
 & & \delta'_2 & &
\end{array}
$$
of $\mathscr{D}_{\text{DPOm}}$.

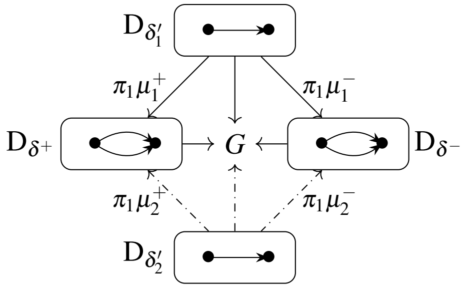
As stated above we will use the partial transformations that are accessible from $\mathscr{D}|_G^{\mathscr{S}}$ through $\mathsf{P} \circ \mathsf{I}_G \circ \mathsf{I}_{\mathscr{S}}$ (a restriction of P). We first need to build an interface (often called *context* by analogy with term rewriting) between the input G and the expected output H. In Parallel Coherent Transformation [2] the context is obtained as a limit of the morphisms $\mathrm{f}_\delta : \mathrm{D}_\delta \to G$ for all δ in a set Δ of direct transformations, hence of a diagram that is a sink to G and thus corresponds to a discrete diagram in $\mathscr{C} \setminus G$. In Global Coherent Transformations the context (denoted C_Δ below) is obtained similarly, but now Δ is a category and the diagram contains the morphisms $\pi_1\mu : \mathrm{D}_{\delta'} \to \mathrm{D}_\delta$ for all $\mu : \delta \to \delta'$ in Δ.

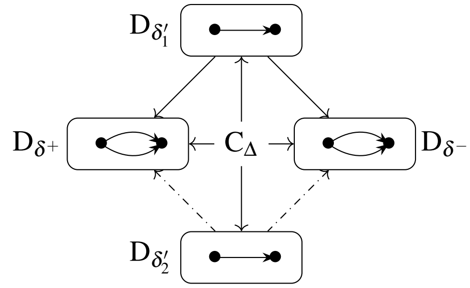
Definition 4.4 (functor $\mathsf{P}^{\leftarrow}_\Delta$, limit $f_\Delta : \mathrm{C}_\Delta \to G$, limit cone γ_Δ)**.** For any subcategory Δ of $\mathscr{D}|_G^{\mathscr{S}}$ let $\mathsf{P}^{\leftarrow}_\Delta : \Delta^{\text{op}} \to \mathscr{C} \setminus G$ be the contravariant functor that maps every $\delta \in \Delta$ to $\mathrm{f}_\delta : \mathrm{D}_\delta \to G$ and every morphism μ of Δ to $\pi_1\mu$. Let $f_\Delta : \mathrm{C}_\Delta \to G$ be the limit of $\mathsf{P}^{\leftarrow}_\Delta$ and γ_Δ be the limit cone from f_Δ to $\mathsf{P}^{\leftarrow}_\Delta$.

Note that if Δ is empty then the limit f_Δ of the empty diagram is the terminal object of $\mathscr{C} \setminus G$, that is 1_G, hence $\mathrm{C}_\Delta = G$.

Example 4.5. Let $\Delta = \mathscr{D}_{\text{DPOm}}|_G^{\mathscr{S}}$. The diagram on the left below corresponds to the functor $\mathsf{P}^{\leftarrow}_\Delta$ together with the morphisms $\mathrm{f}_{\delta_i^\pm} : \mathrm{D}_{\delta_i^\pm} \to G$ (objects in $\mathscr{C} \setminus G$). The limit of this diagram yields $\mathrm{C}_\Delta = \bullet \quad \bullet$ and the limit cone is represented on the right.

[2] We consider transformations only up to isomorphisms, see Footnote 3.



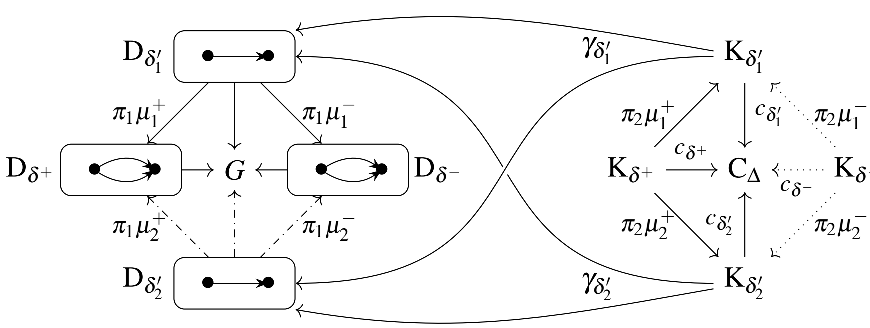


We next need to check that the transformations in Δ do not conflict with each other, i.e., that for all $\delta \in \Delta$ the image of K_δ in G is not only preserved by δ (in D_δ) but also by all other transformations $\delta' \in \Delta$. This is ensured by finding (natural) cones from these K_δ to the $D_{\delta'}$, which we shall formulate with $\mathsf{P}^{\leftarrow}_{\Delta}$, hence in $\mathscr{C} \setminus G$.

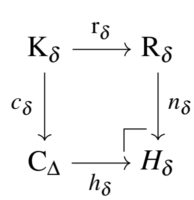
Definition 4.6 (coherent system of cones, morphisms c_δ, global coherence)**.** A *coherent system of cones for* Δ is a set of cones γ_δ from $f_\delta \circ k_\delta$ to $\mathsf{P}^{\leftarrow}_{\Delta}$ such that $\gamma_\delta \delta = k_\delta$ for all $\delta \in \Delta$, and $\gamma_\delta = \gamma_{\delta'} \circ \pi_2\mu$ for all $\mu : \delta \to \delta'$ in Δ. Δ is *globally coherent* if there exists a coherent system of cones for Δ. We then let $c_\delta : f_\delta \circ k_\delta \to f_\Delta$ be the unique morphism in $\mathscr{C} \setminus G$ such that $\gamma_\delta = \gamma_\Delta \circ c_\delta$.

Note that if $\gamma_{\delta'}$ is a cone from $f_{\delta'} \circ k_{\delta'}$ to $\mathsf{P}^{\leftarrow}_{\Delta}$ then $\gamma_{\delta'} \circ \pi_2\mu$ is a cone from $f_\delta \circ k_\delta$ to $\mathsf{P}^{\leftarrow}_{\Delta}$, hence global coherence means that we should find cones for the direct transformations (say δ'_1 and δ'_2) from the top rules, with the constraint that they should be compatible on common subtransformations $\delta'_1 \leftarrow \delta \to \delta'_2$. If $\mathscr{S}$ and therefore Δ are discrete, this amounts to parallel coherence, see [2].

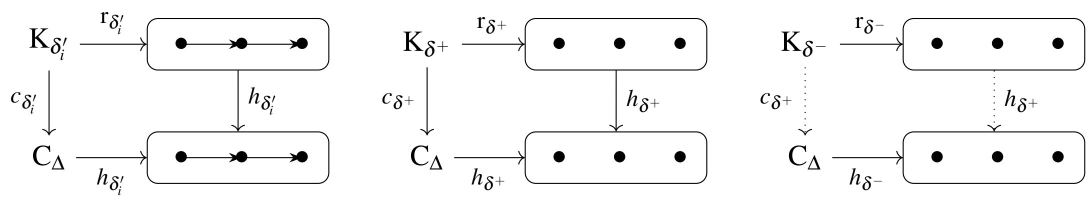
Example 4.7. On our example the four graphs $K_{\delta_i^\pm}$ are equal to $\bullet \quad \bullet$. It is easy to build the four cones from the four morphisms from $K_{\delta'_i}$ to $D_{\delta'_i}$ depicted below, by composing them with the $\pi_1\mu_i^\pm$ on the left and the $\pi_2\mu_i^\pm$ on the right. On the right are also depicted the morphisms $c_{\delta_i^\pm}$.



Definition 4.8 (morphisms $h_\delta : C_\Delta \to H_\delta$)**.** If Δ is globally coherent for all $\delta \in \Delta$ then c_δ can be viewed as a $\mathscr{C}$-morphism $c_\delta : K_\delta \to C_\Delta$, and we consider the following pushout in $\mathscr{C}$.

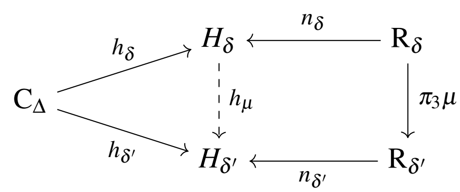


Example 4.9. On our example we get:



We now turn h into a functor.

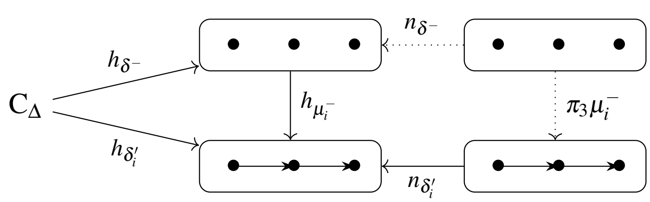
Proposition 4.10. *For every $\mu : \delta \to \delta'$ in Δ there exists a unique h_μ such that*



commutes.

Corollary 4.11. *By unicity we get $h_{\mu' \circ \mu} = h_{\mu'} \circ h_\mu$.*

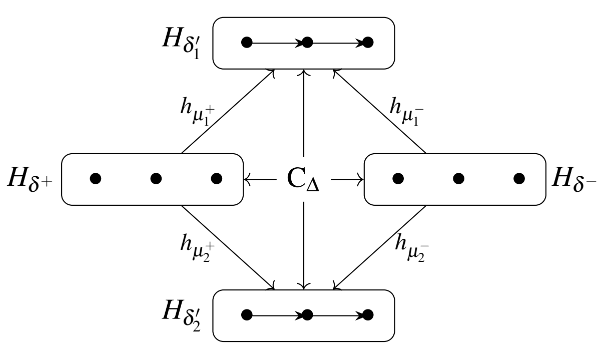
Example 4.12. For instance the morphisms $\mu_i^- : \delta^- \to \delta_i'$ yield the morphisms $h_{\mu_i^-}$ depicted below.



Definition 4.13 (functor $\mathsf{P}_\Delta^{\to}$, colimit $h_\Delta : \mathrm{C}_\Delta \to H_\Delta$)**.** If Δ is globally coherent let $\mathsf{P}_\Delta^{\to} : \Delta \to \mathrm{C}_\Delta \backslash \mathscr{C}$ be the functor defined by $\mathsf{P}_\Delta^{\to}\delta = h_\delta$ (interpreted as an object of $\mathrm{C}_\Delta \backslash \mathscr{C}$) and $\mathsf{P}_\Delta^{\to}\mu = h_\mu$ for all $\mu : \delta \to \delta'$ in Δ. Let $h_\Delta : \mathrm{C}_\Delta \to H_\Delta$ be the colimit[3] of $\mathsf{P}_\Delta^{\to}$, then the $\mathscr{C}$-span $G \xleftarrow{f_\Delta} \mathrm{C}_\Delta \xrightarrow{h_\Delta} H_\Delta$ is a *Global Coherent Transformation by* Δ.

If Δ is empty then the colimit h_Δ of the empty diagram is the initial object of $\mathrm{C}_\Delta \backslash \mathscr{C}$, that is 1_{C_Δ}, hence $H_\Delta = \mathrm{C}_\Delta = G$. Generally, the functor $\mathsf{P}_\Delta^{\to}$ depends on the choice of cones γ_δ for $\delta \in \Delta$, hence h_Δ is not determined by Δ.

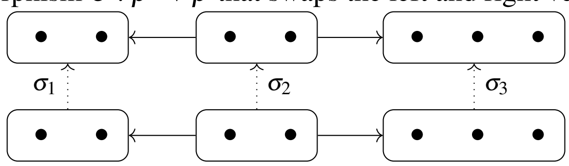
Example 4.14. The functor $\mathsf{P}_\Delta^{\to}$ applied to Δ yields the following diagram



[3] Global Coherent Transformations are obtained as limits and colimits of diagrams whose index category is Δ, hence are not affected by isomorphisms in Δ, which can therefore be replaced by its skeleton.

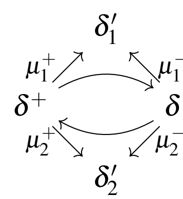
The leftmost vertices of these five graphs are connected as images or preimages of each other, and similarly for the five right vertices, and the four middle vertices. The four edges are not likewise connected, hence the colimit of this diagram is the expected result $H =$  . We therefore see that the two middle vertices created in δ'_1 and δ'_2 are merged by their common subtransformation δ^+ (or δ^-), but also that the two middle vertices created in δ^+ and δ^- are merged by their common upper transformation δ'_1 (or δ'_2).

If we apply $\mathscr{S}$ to the graph $G' =$ then rule ρ' does not apply to G' and hence the two matchings of ρ in G' apply independently, thus adding two vertices to G'. We can merge them by adding to $\mathscr{S}$ the following rule morphism $\sigma : \rho \to \rho$ that swaps the left and right vertices:



We have $\sigma^2 = 1_\rho$ hence σ is an automorphism of ρ. Adding σ to $\mathscr{S}$ means that ρ will be applied modulo automorphisms; this generalizes to the algebraic context the notion of Parallel Rewriting Modulo Automorphism devised in an algorithmic approach in [1].

Since $\sigma^+ \circ \sigma = \sigma^-$ and $\sigma^- \circ \sigma = \sigma^+$, the new rule system is $\mathscr{S}' = \sigma \circlearrowleft \rho \underset{\sigma^-}{\overset{\sigma^+}{\rightrightarrows}} \rho'$. If we apply $\mathscr{S}'$ to G, we add two new morphisms in $\mathscr{D}_{\mathrm{DPOm}}|^{\mathscr{S}}_{G}$, i.e, $\Delta' = \mathscr{D}_{\mathrm{DPOm}}|^{\mathscr{S}'}_{G} =$



. It is easy to see that the Global Coherent Transformation by Δ' is the same as above with Δ.

We finally prove that, apart from this mechanism of sharing common transformations, isolated transformations always subsume their subtransformations, so that morphisms in $\mathscr{R}$ are rule subsumptions as intended.

Proposition 4.15. *If Δ' is restricted to δ' and Δ to $\mu : \delta \to \delta'$ (or more generally if δ' is terminal in Δ) then Δ and Δ' are globally coherent and $H_\Delta \simeq H_{\Delta'}$.*

5 Rewriting Environments and Their Properties

Definitions 3.2, 3.7 and 4.1 provide two Rewriting Environments that we may call DPO and DPOm. By Proposition 3.6 it is obvious that $\mathsf{R}_{\mathrm{DPO}}$ and $\mathsf{R}_{\mathrm{DPOm}}$ are faithful when $\mathscr{C}$ is the category of graphs. This is easily seen to generalize to all adhesive categories [12]. In fact, we can generalize Proposition 3.6 as follows:

Proposition 5.1. *If $\mathscr{C}$ is adhesive, $\delta, \delta' \in \mathscr{D}_{\mathrm{DPO}}$ and $\sigma : \mathsf{R}_{\mathrm{DPO}}\delta \to \mathsf{R}_{\mathrm{DPO}}\delta'$ such that $\delta\mathrm{m} = \delta'\mathrm{m} \circ \sigma_1$ then there exists a unique $\mu : \delta \to \delta'$ such that $\mathsf{R}_{\mathrm{DPO}}\mu = \sigma$.*

A property that one might reasonably expect is that when a rule applies and yields a direct transformation then its subrules also apply and yield subtransformations. We express this by means of the following notion.

Definition 5.2 (right-full)**.** A functor $\mathsf{F} : \mathscr{A} \to \mathscr{B}$ is *right-full*[4] if for all $a' \in \mathscr{A}$, all $b \in \mathscr{B}$ and all

[4]This is named after the symmetric definition of *left-full* functors in [18, p. 63].

$g : b \to Fa'$, there exist $a \in \mathscr{A}$ and $f : a \to a'$ such that $Ff = g$.

It is obvious that right-fullness is closed by composition.

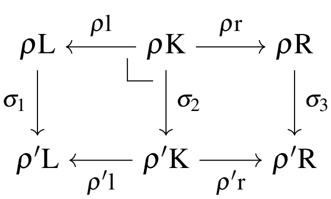
Lemma 5.3. *I_G is a right-full embedding.*

Proposition 5.4. *If R is right-full (resp. faithful) then so is $\mathsf{R}|_G^{\mathscr{S}}$ for every rule system $\mathscr{S}$ and $G \in \mathscr{C}$.*

Hence when R is right-full and faithful every morphism $\sigma : \rho \to \rho'$ in $\mathscr{S}$ is reflected by a morphism in $\mathscr{D}|_G^{\mathscr{S}}$ whenever ρ' is reflected by a direct transformation δ' (i.e., whenever ρ' applies to G), and this morphism is uniquely determined by σ and δ'. According to Proposition 3.4 it is obvious that $\mathsf{R}_{\mathrm{DPOm}}$ is right-full (when $\mathscr{C}$ is the category of graphs). It is easy to see that $\mathsf{R}_{\mathrm{DPO}}$ is not right-full (since σ_1 may not be monic).

We now consider the case of Sesqui-Pushouts [5]. It is based on the notion of final pullback complement that allows not only to remove parts of the input G but also to make copies of parts of G (when $\rho\mathrm{l}$ below is not monic).

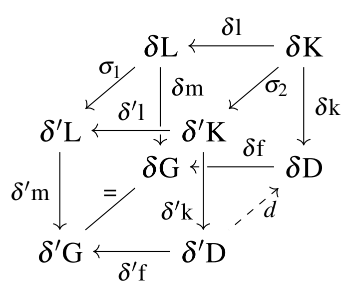
Definition 5.5. For any category $\mathscr{C}$, let $\mathscr{R}_{\mathrm{SqPO}}$ be the category whose objects are the functors $\rho : \mathbf{sp} \to \mathscr{C}$ and morphisms $\sigma : \rho \to \rho'$ are triples $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ of $\mathscr{C}$-morphisms such that



commutes in $\mathscr{C}$ and the left square is a pullback, with obvious composition and identities. Let $\mathscr{R}_{\mathrm{SqPOm}}$ be the subcategory with morphisms σ such that σ_1 and σ_2 are monics.

A *direct SqPO-transformation* in $\mathscr{C}$ is a functor $\delta : \mathbf{dt} \to \mathscr{C}$ such that $\delta\mathrm{f}$,$\delta\mathrm{k}$ is a final pullback complement of $\delta\mathrm{m}$, $\delta\mathrm{l}$, and $(\delta\mathrm{r}, \delta\mathrm{n}, \delta\mathrm{k}, \delta\mathrm{g})$ is a pushout.

Proposition 5.6. *For every direct SqPO-transformations δ, δ' with corresponding SqPO-rules ρ, ρ', every $\sigma : \rho \to \rho'$ in $\mathscr{R}_{\mathrm{SqPO}}$ such that $\delta\mathrm{m} = \delta'\mathrm{m} \circ \sigma_1$, there exists a unique $\mathscr{C}$-morphism d such that*



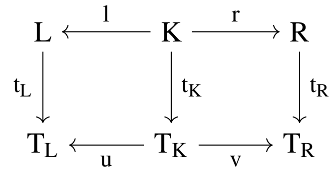
commutes.

Here the existence of d means not only that ρ' removes at least as much as its subrule ρ, but also that it makes at least as many copies of the items of G. It is then easy to define the category $\mathscr{D}_{\mathrm{SqPO}}$ of direct SqPO-transformations, the category $\mathscr{D}_{\mathrm{SqPOm}}$ of direct SqPO-transformations with monic matches and faithful functors $\mathsf{R}_{\mathrm{SqPO}} : \mathscr{D}_{\mathrm{SqPO}} \to \mathscr{R}_{\mathrm{SqPO}}$ and $\mathsf{R}_{\mathrm{SqPOm}} : \mathscr{D}_{\mathrm{SqPOm}} \to \mathscr{R}_{\mathrm{SqPOm}}$, as in Definition 3.7. We leave this to the reader.

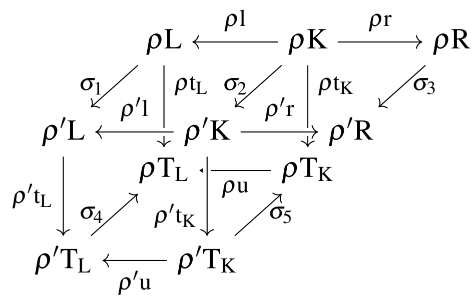
Proposition 5.7. *In the category of graphs $\mathsf{R}_{\mathrm{SqPOm}}$ is right-full.*

We next consider the case of PBPO-rules [4], that also enables copies of parts of G but with better control of the way they are linked together and to the rest of G. The drawback is that matchings of the left-hand side of a rule into G should be completed with a co-match from G to a given type of the left-hand side.

Definition 5.8 (category $\mathscr{D}_{\mathrm{PBPO}}$, direct PBPO-transformations)**.** Let **pb** be the category generated by the graph

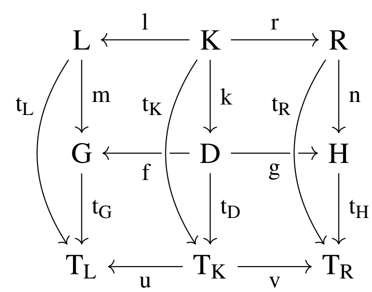


with relations $\mathrm{t_L} \circ \mathrm{l} = \mathrm{u} \circ \mathrm{t_K}$ and $\mathrm{t_R} \circ \mathrm{r} = \mathrm{v} \circ \mathrm{t_K}$. A *PBPO-rule* in $\mathscr{C}$ is a functor $\rho : \mathbf{pb} \to \mathscr{C}$. A morphism $\sigma : \rho \to \rho'$ is a 5-tuple $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5)$ of $\mathscr{C}$-morphisms such that



commutes. Let $\mathscr{D}_{\mathrm{PBPO}}$ be the category of morphisms of PBPO-rules on $\mathscr{C}$, with the obvious composition and identities.

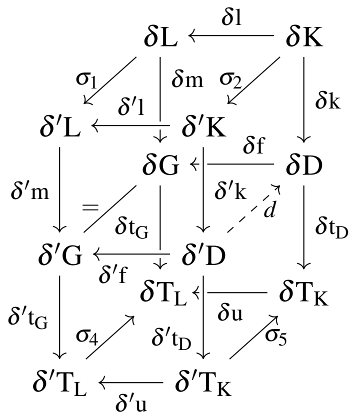
Let **pbt** be the category generated by



with all commuting relations, a *direct PBPO-transformation* in $\mathscr{C}$ is a functor $\delta : \mathbf{pbt} \to \mathscr{C}$ such that $(\delta\mathrm{f}, \delta\mathrm{t_G}, \delta\mathrm{t_D}, \delta\mathrm{u})$ is a pullback and $(\delta\mathrm{r}, \delta\mathrm{n}, \delta\mathrm{k}, \delta\mathrm{g})$ is a pushout.

To every direct PBPO-transformation obviously corresponds a PBPO-rule and a partial transformation.

Proposition 5.9. *For every direct PBPO-transformations δ, δ' with corresponding PBPO-rules ρ, ρ', every $\sigma : \rho \to \rho'$ in $\mathscr{D}_{\mathrm{PBPO}}$ such that $\delta\mathrm{m} = \delta'\mathrm{m} \circ \sigma_1$ and $\delta\mathrm{t_G} = \sigma_4 \circ \delta'\mathrm{t_G}$, there exists a unique $\mathscr{C}$-morphism d such that*



commutes.

We leave it to the reader to define a Rewriting Environment for PBPO-rules and transformations, with a right-full faithful functor $\mathsf{R}_{\mathrm{PBPO}} : \mathscr{D}_{\mathrm{PBPO}} \to \mathscr{R}_{\mathrm{PBPO}}$ (provided $\mathscr{C}$ has pushouts and pullbacks).

We finally observe that if $\mathscr{R}_1 \xleftarrow{\mathsf{R}_1} \mathscr{D}_1 \xrightarrow{\mathsf{P}_1} \mathscr{C}_{\mathrm{pt}}$ and $\mathscr{R}_2 \xleftarrow{\mathsf{R}_2} \mathscr{D}_2 \xrightarrow{\mathsf{P}_2} \mathscr{C}_{\mathrm{pt}}$ are Rewriting Environments then so is $\mathscr{R}_1 + \mathscr{R}_2 \xleftarrow{\mathsf{R}_1+\mathsf{R}_2} \mathscr{D}_1 + \mathscr{D}_2 \xrightarrow{[\mathsf{P}_1,\mathsf{P}_2]} \mathscr{C}_{\mathrm{pt}}$. It is therefore possible to mix rules of different approaches to transform a graph, though of course rules of distinct approaches cannot subsume each other.

6 Conclusion and Future Work

The general notion of a rule subsumption is given through Rewriting Environments, where abstract categories of rules and direct transformations are related to a specific category of partial transformations $\mathscr{C}_{\mathrm{pt}}$. Global Coherent Transformation are built from partial transformations in a way pertaining both to Parallel Coherent Transformation, by the use of limits on local contexts, and to Global Transformations, by applying categories of rules.

Other notions of subrules exist in Graph Transformations. A common notion comes from representing sequences of transformations as a single transformation (horizontal composition); this is relevant to sequential rewriting. Another notion is known as vertical composition (or rule instantiation): the result of applying a rule to a graph is reified as a rule that can be applied to a new graph (these two notions can be mixed as in [15]). This appears to be a particular case of subsumption where the induced direct transformations are parallel (same context, same result).

We have provided Rewriting Environments for the most common approaches to algebraic rewriting, except the Single Pushout [14], which will be done in a forthcoming paper. We also intend to show that Global Transformations can be obtained by a suitable environment (except when Δ is empty). We also need to further analyze the properties of the DPO Rewriting Environment: when $\mathscr{C}$ is an adhesive category it is an open question whether $\mathsf{R}_{\mathrm{DPOm}}$ is right-full.

It may seem strange that, through $\mathscr{C}_{\mathrm{pt}}$, rules are not assumed to have left-hand sides and direct transformations are not assumed to use matchings. The notion of Rewriting Environment is as simple as required to define Global Coherent Transformations, but does not guarantee some properties that the user might reasonably expect. In particular it does not prevent the categories $\mathscr{R}$ and $\mathscr{D}$ from being discrete (but this is correct if no subsumption is possible). In the future we will enhance Rewriting Environments with a notion of matching in order to better understand their structure.

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