Partial Markov Categories - Extended Abstract

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***Abstract*—Partial Markov categories are an axiomatisation of Markov categories with partiality. Partiality allows us to express observations and constraints, but the extension of Markov categories to the partial setting requires some design choices, like imposing an extra axiom on conditionals. We present partial Markov categories and investigate some conceptual consequences of these design choices that extend our previous work: "Evidential Decision Theory via Partial Markov Categories" accepted for publication at the symposium on Logic in Computer Science 2023 (LiCS'23).**

1 Partial Markov categories

Partial Markov categories [DR23] are an axiomatisation of Markov categories [Fri20] with partiality. Like Markov categories, partial Markov categories are symmetric monoidal categories with a copy-discard structure [CG99] and conditionals (Figure 1, [Fri20]). However, morphisms are not required to be total: discarding the output of a morphism may still have an effect and differ from just discarding its input.

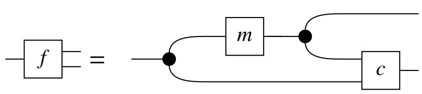
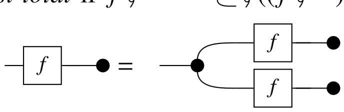


Fig. 1: Conditionals require that a stochastic process f be split into a marginal m and a conditional c. In Markov categories, this is equivalent to the definition in [Fri20].

Following the interpretation of morphisms in Markov categories, morphisms in partial Markov categories represent stochastic processes that may "fail". Partiality increases the expressive power of a theory of stochastic processes, allowing to express observations and constraints (Section 2), but this extension also requires some design choices. Conditionals in Markov categories are, like all other morphisms, total. This ensures some convenient properties about marginals: the marginal of $f\colon X \to A \otimes B$ on A is obtained by discarding the B output, $f \fatsemi (\mathsf{id} \otimes -\!\bullet)$. In order to keep some of these convenient properties we impose an extra condition on conditionals: we ask them to be *quasi-total*.

Definition 1.1. A morphism $f\colon X \to Y$ in a copy-discard category is *quasi-total* if $f \fatsemi -\!\bullet = -\!\bullet\!\!< \fatsemi ((f \fatsemi -\!\bullet) \otimes (f \fatsemi -\!\bullet))$.



Definition 1.2 ([DR23])**.** A *partial Markov category* is a copy-discard category with quasi-total conditionals.

In cartesian restriction categories [CL02; CL03; CL07] the domain of definition of a morphism f is defined as $(f \fatsemi -\!\bullet)$: discarding the output of f gives the inputs on which f does not fail. In partial Markov categories, discarding the output of a morphism, $(f \fatsemi -\!\bullet)$, corresponds to the *probability of failure* of f. With this interpretation, quasi-totality means determinism of the probability of failure, i.e. $(f \fatsemi -\!\bullet)$ can be copied: on each input, the channel can either certainly fail, or never fail and return a total distribution. The inputs on which the channel never fails are its *domain of definition*.

Despite this interpretation, the definition of quasi-totality still requires some justification. If we asked conditionals to be total, like in Markov categories, this would be too strong because they would fail to exist even in the simplest example of finitary subdistributions (Example 1.4). On the other hand, leaving conditionals unrestricted would prevent us from having some properties that they have in Markov categories. Quasi-totality seems a sweet spot, and it allows us to prove that marginals are given by discarding one of the inputs, as in Markov categories (see Proposition 3.14 in [DR23]).

Example 1.3. Markov categories with conditionals are exactly partial Markov categories where all maps are total.

Example 1.4. The Kleisli category of the finitary subdistribution monad, $\mathsf{Kl}(\mathbf{D}_{\leq 1})$, is a partial Markov category. Given a morphism $f\colon X \to A \otimes B$, its *marginal* on A is

$$m(a \mid x) = \sum_{b \in B} f(a, b \mid x) \quad \text{and} \quad m(\perp \mid x) = f(\perp \mid x)\ ,$$

and a *conditional* with respect to A is

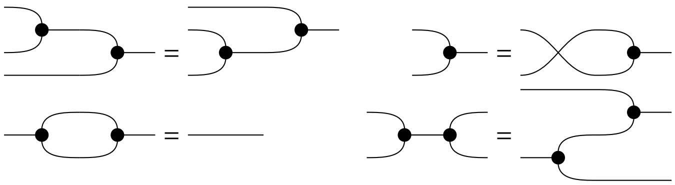
$$c(b \mid a, x) = \begin{cases} \dfrac{f(a,b \mid x)}{m(a \mid x)}, & \text{if } m(a \mid x) \neq 0; \\ 0, & \text{otherwise;} \end{cases}$$

$$c(\perp \mid a, x) = \begin{cases} 0, & \text{if } m(a \mid x) \neq 0; \\ 1, & \text{otherwise.} \end{cases}$$

Example 1.5. The category of continuous subdistributions on standard Borel spaces, $\mathsf{BorelStoch}_{\leq 1}$, is a partial Markov category [DR23].

Some partial Markov categories have an extra piece of structure that may be interpreted as equality checks. We call these *discrete partial Markov categories* following discrete cartesian restriction categories [CGH12; Di +21].

Definition 1.6. A copy-discard category C has *comparators* if every object X has a morphism, $\supset\!\!\bullet\!-_X\colon X \otimes X \to X$, that is uniform, commutative, associative and satisfies the partial Frobenius axioms below with the copy-discard structure.



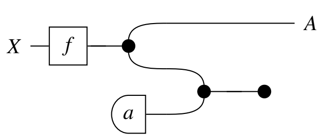
Definition 1.7. A *discrete partial Markov category* is a copy-discard category with conditionals and comparators. In other words, it is a partial Markov category with comparators.

Example 1.8. The Kleisli category of the finitary subdistribution monad, $\mathsf{Kl}(\mathbf{D}_{\leq 1})$, is a discrete partial Markov category. The comparator $\succ\!\!\bullet\!\!-_X \colon X \otimes X \to X$ is given by

$$\succ\!\!\bullet\!\!-_X(x \mid x_1, x_2) = \begin{cases} 1, & \text{if } x = x_1 = x_2; \\ 0, & \text{otherwise.} \end{cases}$$

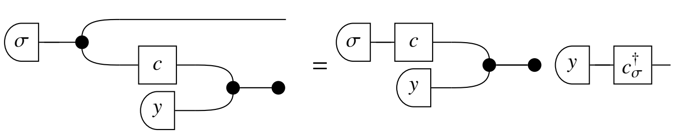
2 Updates and observations via constraints

The structure and axioms of discrete partial Markov categories can express the process of updating a stochastic model on observations. Updating on an observation $a \colon I \to A$ means to restrict the model $f \colon X \to A$ to scenarios that are compatible with this observation.

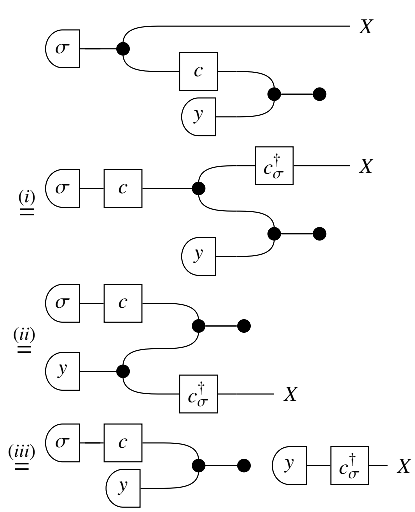


When the observation is deterministic, Bayes' theorem applies.

Theorem 2.1 (Synthetic Bayes, Theorem 3.28 in [DR23])**.** *In a discrete partial Markov category, observing a deterministic $y \colon I \to Y$ from a prior distribution $\sigma \colon I \to X$ through a channel $c \colon X \to Y$ is the same, up to scalar, as evaluating the Bayesian inversion of the channel $c^{\dagger}_{\sigma}(y)$.*



Proof. We employ string diagrams.



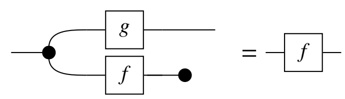
The equalities follow from: (*i*) quasi-total conditionals, (*ii*) the partial Frobenius axioms (Definition 1.6), and (*iii*) determinism of the observation y. □

All the structure and axioms of discrete partial Markov categories are necessary for this result: the comparator structure for constraining the model to be compatible with the observation and conditionals for propagating this constraint to the prior and update it.

3 Towards unique conditionals

If a Markov category has unique conditionals, then it is a preorder (Proposition 11.15 in [Fri20]). A similar collapse, although less extreme, happens for partial Markov categories (Proposition 3.15 in [DR23]). However, among all conditionals in $\mathsf{Kl}(\mathbf{D}_{\leq 1})$, one has a privileged position: the one defined in Example 1.4. The only arbitrary choice in that definition is the value of $c(- \mid a, x)$ when $m(a, x) = 0$, as we could have chosen any subdistribution on B. Choosing the subdistribution that always fails gives a conditional with "minimal" information: whenever there is an arbitrary choice to be made, it fails. Can we characterise these "minimal" conditionals? They are the minimal elements of a preorder on quasi-total morphisms.

Definition 3.1. For quasi-total morphisms $f, g \colon X \to A$ in a partial Markov category, we say that f is a *restriction* of g, $f \leq g$, if $-\!\!\bullet\!\!\prec \,⨟\, (g \otimes (f \,⨟\, -\!\!\bullet)) = f$.



This defines a preorder on the quasi-total morphisms.

This preorder has been defined in cartesian restriction categories [CL02], where its interpretation is that $f \leq g$ whenever f is a restriction of g on a smaller domain. In partial Markov categories, this is not a preorder on all morphisms because it is not reflexive in general: $f \leq f$ if and only if f is quasi-total. This is a consequence of Proposition 3.5 in [DR23].

Definition 3.2. A partial Markov category has *minimal conditionals* whenever, for every $f \colon X \to A \otimes B$, the preorder $\leq$ on its quasi-total conditionals has a minimal element.

Example 3.3. The Kleisli category of the finitary subdistribution monad $\mathsf{Kl}(\mathbf{D}_{\leq 1})$ has minimal conditionals.

4 Future directions

We have defined a preorder on the set of conditionals of morphisms in partial Markov categories. When this preorder has a minimal element, it gives a canonical choice for conditionals. This happens in the Kleisli category of the subdistribution monad, but more sophisticated examples, like $\mathsf{BorelStoch}_{\leq 1}$, are left for further work. The conditions that ensure the existence of minimal elements of this preorder also remain to be explored.

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