Polycategories of Supermaps on Monoidal Categories

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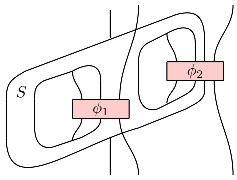
Based on the pre-print *Free Polycategories for Unitary Supermaps of Arbitrary Dimension*.

Introduction and Context: A common emergent theme within quantum information theory, theoretical computer science, and applied category theory, has been the development of the concept of a hole into-which parts of processes can be inserted as depicted in the following intuitive picture:



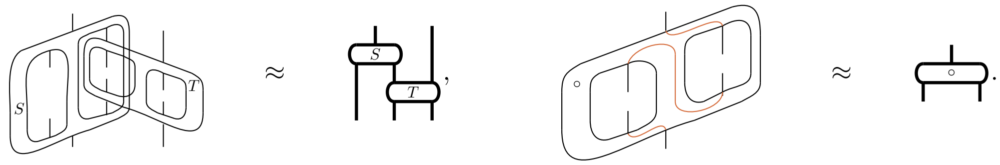
Such instances can be seen within the study of quantum information processing [1–7], quantum foundations [1–5, 8, 9], bidirectional programming [10–12], game-theory [13–16], machine learning [17], open systems dynamics [18], and financial trading [19]. In all but the quantum case, the study of such a concept has been restricted to the case of circuit diagrams into-which holes have been punctured called combs. However, as discovered in quantum contexts, there are natural notions of hole-into-which a process can be inserted which are not representable as circuit diagrams into which holes have been punctured [20]. This is most clearly seen by considering the probabilistic combination or quantum superposition of two such diagrams-with-holes which in either case puts those holes into an indefinite compositional structure. Referred to in the quantum literature as *supermaps* [2–4], also known as *process matrices* [8], such holes are often interpreted as regions without a predetermined causal configuration. The development of the framework of quantum supermaps has been used to formalise general theories of quantum devices as resources, and given a broad setting in which the computational and information-processing advantages of quantum causal structures can be studied [3, 21–28].

The motivation for this paper is to contribute towards the development of the formalisation of supermaps on general monoidal categories [7, 10, 29, 40]. Within physics the lack of such a generalisation is a fundamental problem, a broad area of research into the foundations of physics is that of the properties of operational (generalised) probabilistic theories [30–32], these are frameworks within which physical theories can be studied in terms of their information processing capabilities. Without a generalised approach to supermaps, we are without a framework for the study of stronger-than-quantum correlations between causal structures. Perhaps more urgently, the answer to the question of the appropriate definition of supermaps on arbitrary Hilbert spaces is currently unclear [4, 9, 29].

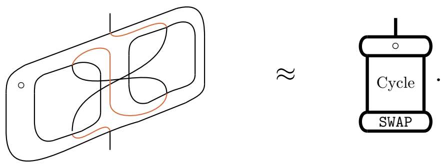
Aims of The Manuscript: The concrete goal of this paper is to define a construction which sends any symmetric monoidal category to a theory of supermaps on that monoidal category. To be satisfied with such a construction we must have some goal-post features in mind that we would expect to see.

Physical Features: There are some established definitions of supermaps which we would like our constructions to recover. More concretely, superchannels [2] are typically accepted as the right notion of supermap on the quantum channels, and superunitaries [33, 34] are typically accepted as the right notion of supermap on the unitaries.

Compositional Features: We consider two categorical aspects of theories of supermaps; composition rules for insertion of input/output pairs into holes, and existence of supermaps which implement the composition rules of the category they act on. Each of these aspects can be visualised by trading informal pictures of supermaps with more formal circuit-diagrammatic ones in which holes are drawn at the bottom, and input/output pairs are drawn at the top:



For the former aspect, composition along two or more formal (bold) wires is however forbidden within these models, meaning that supermaps of this shape form not monoidal categories but instead form a more restricted notion of *polycategory* [35]. Informally polycategories are theories of processes with multiple inputs and outputs, but with composition permitted only along one wire at a time. The latter aspect, is the familiar categorical notion of enrichment, identified previously as a core feature of theories of supermaps [36]. It is enrichment which forbids multi-wire composition, since when combined such concepts would result in the production of pathological time-loops as can be seen in this simple intuitive example:



The main contribution of this paper is to develop a construction which assigns to each symmetric monoidal category a model of black-box supermaps with the above features. Models exist which either; are black-box but not applicable to all symmetric monoidal categories [37–39], or instead are applicable to all symmetric monoidal categories but without being black-box enough to incorporate quantum causal structures [7].

Methods and Results: In our approach we build on a recently introduced notion of locally-applicable transformation [29]. A locally-applicable transformation of type $S : [A, A'] \longrightarrow [B, B']$ on a symmetric monoidal category $\mathbf{C}$ is a family of functions $S_{X,X'} : \mathbf{C}(A \otimes X, A' \otimes X') \to \mathbf{C}(B \otimes X, B' \otimes X')$ such that for all f, g then $S_{X,X'}((i \otimes g) \circ (\phi \otimes i) \circ (i \otimes f)) = (i \otimes g) \circ (S_{Y,Y'}(\phi) \otimes i) \circ (i \otimes f)$. For our construction we consider *slots*, which are families of functions so local that they commute with all locally-applicable transformations, meaning that they can be formalised as central in the premonoidal category of locally-applicable transformations. After modelling one-input supermaps with slots, we use them to inductively define multi-input *polyslots*. As an application we show that quantum superpositions of circuits-with-holes on arbitrary quantum systems, are indeed polyslots on the category of unitaries between general (even non-seperable) Hilbert spaces.

In our first theorem we show how to compose polyslots to form a polycategory $\mathbf{pslot}[\mathbf{C}]$ which enriches the monoidal structure of $\mathbf{C}$.

Theorem 1. *The polyslots on* $\mathbf{C}$ *define a symmetric polycategory* $\mathbf{pslot}[\mathbf{C}]$.

We then turn to recovering superchannels and superunitaries. Concretely, letting $\mathbf{U}, \mathbf{QC}$ be the categories of unitaries and quantum channels, and letting $\mathbf{uQS}, \mathbf{QS}$ be the symmetric polycategories of superunitaries and superchannels respectively:

Theorem 2. $\mathbf{pslot}[\mathbf{U}]$ *is equivalent as a symmetric polycategory to* $\mathbf{uQS}$*, and* $\mathbf{pslot}[\mathbf{QC}]$ *is equivalent as a symmetric polycategory to* $\mathbf{QS}$*.*

To summarise, the paper provides a construction $\mathbf{pslot}[\mathbf{C}]$ for supermaps on a symmetric monoidal category $\mathbf{C}$, which has all of our proposed physical and compositional features, and furthermore is broad enough to model quantum causal structures on arbitrary Hilbert spaces.

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