Data-Parallel Algorithms for String Diagrams

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We give parallel algorithms for string diagrams represented as *structured cospans of ACSets*. Specifically, we give *linear* (sequential) and *logarithmic* (parallel) time algorithms for composition, tensor product, construction of diagrams from arbitrary Σ-terms, and application of functors to diagrams.

Our datastructure can represent morphisms of both the free symmetric monoidal category over an arbitrary signature as well as those with a chosen Special Frobenius structure. We show how this additional (hypergraph) structure can be used to map diagrams to *diagrams of optics*. This leads to a case study in which we define an algorithm for efficiently computing symbolic representations of gradient-based learners based on reverse derivatives.

The work we present here is intended to be useful as a *general purpose* datastructure. Implementation requires only *integer arrays* and well-known algorithms, and is data-parallel by constuction. We therefore expect it to be applicable to a wide variety of settings, including embedded and parallel hardware and low-level languages.

1 Introduction

String diagrams are a formal graphical syntax [13] for representing morphisms of monoidal categories which is now widely used (see for example [7, 8, 9, 2]). The purpose of this paper is to make string diagrams not just a convenient notation for algebraic reasoning, but also an efficient general-purpose tool in computing with graphical structures in a compositional manner. To that end, the datastructures and algorithms we define satisfy the following desiderata.

Fast and data-parallel. Our algorithms are data-parallel by construction, and have *linear* (sequential) and *logarithmic* (parallel) time complexities.

Minimal primitives. Our datastructures are defined in terms of simple integer arrays. Moreover, we assume only a small number of simple, well-known primitive operations (e.g., prefix sum). This makes it possible to implement our algorithms in a wide variety of settings, such as embedded and parallel (i.e., GPU) hardware.

Simple to implement correctly. Key parts of our datastructure are defined in terms of the recent construction of *ACSets* [11]. Consequently, implementations are essentially the same as their categorical definitions, making it easier to ensure correctness.

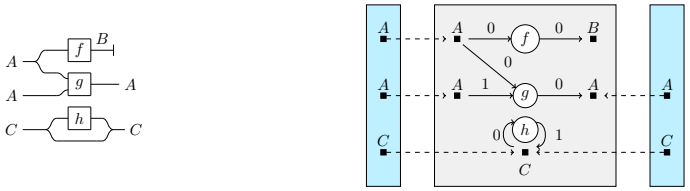
A number of representations of string diagrams such as the wiring diagrams of Catlab [10] have been explored in the literature. Our goals most closely align with the 'hypergraph adjacency representations' of [15]: we aim to make string diagrams useful as a general purpose 'scalable combinatorial syntax'. For example, we hope that our implementation serves as an alternative in cases where a programmer would currently use a tree or directed graph.

However, the primary motivating application for our work is in representing gradient-based learners as optics, as described in [5]. In particular, this motivates perhaps the most significant extension to [15]: our datastructures can 'natively' represent morphisms of the free symmetric monoidal category over a signature with a chosen Special Frobenius monoid.

We now give a brief overview of the main contributions of our paper [16].

1.1 String Diagrams as Structured Cospans of ACSets

Our datastructure represents morphisms of $\mathsf{Free}_{\Sigma+\mathbf{Frob}}$ (the free symmetric monoidal category over the signature $\Sigma+\mathbf{Frob}$) as structured cospans [1] of ACSets [11]. Diagrams represented in this way form a symmetric monoidal category $\mathsf{Diagram}_{\Sigma+\mathbf{Frob}}$ which is isomorphic to $\mathsf{Free}_{\Sigma+\mathbf{Frob}}$. Pictured below is a string diagram in $\mathsf{Free}_{\Sigma+\mathbf{Frob}}$ (left) and its representation as a structured cospan (right).

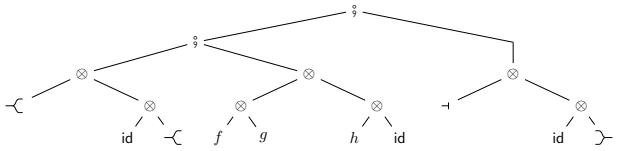


(1)

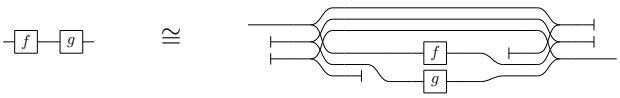
The center box, in grey, depicts a bipartite multigraph modelling the 'internal wiring' of the string diagram, where ■-nodes represent the *wires* of the diagram, and ∘-nodes the *operations*.

1.2 Data-Parallel Algorithms

We give data-parallel algorithms with *linear* (sequential) and *logarithmic* (parallel) time complexity for composition, tensor product, functor application, and for constructing a diagram from a Σ-term. Note that the latter case is required to efficiently construct a diagram from a Σ-term such as the one below, which corresponds to the diagram in (1).



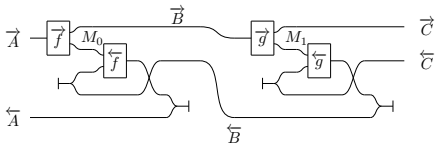
The algorithm given for applying functors to diagrams exploits the hypergraph structure of $\mathsf{Diagram}_{\Sigma+\mathbf{Frob}}$. We define 'Frobenius decompositions' for this purpose (see below) which essentially separate the wires and operations of a diagram.



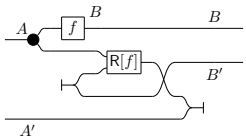
Applying a functor to diagrams of this kind then involves only its action on spiders and operations.

1.3 Diagrams of Optics and Reverse Derivatives

The hypergraph structure of $\mathsf{Diagram}_{\Sigma+\mathbf{Frob}}$ can also be used to efficiently compute composition of *optics* [6, 12], as below.



This allows modeling applications with 'bidirectional information flow' such as [5, 14, 3]. In our specific example, it allows us to define to an efficient algorithm for taking reverse derivatives [4] (see below) and modeling gradient-based learners in general.



Using our algorithm for applying functors to diagrams, this allows us to map diagrams to *diagrams of optics*, and consequently gives an algorithm for taking reverse derivatives of diagrams in linear (sequential) and logarithmic (parallel) time. More generally, this allows for mapping diagrams with *feedback* in terms of the hypergraph structure to *optics with feedback*. That is, systems with notions of both bidirectional information flow and recursion or iteration.

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