Covariant influences for finite discrete dynamical systems

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We develop a rigorous theory of external influences on finite discrete dynamical systems, going beyond the perturbation paradigm, in that the external influence need not be a small contribution. Indeed, the covariance condition can be stated as follows: If we evolve the dynamical system for n time steps and then disturb it, then it is the same as first disturbing the system with the same influence and then letting the system evolve for n time steps. Applying the powerful machinery of resource theories, grounded in category theory, we develop a theory of covariant influences both when there is a purely deterministic evolution and when randomness is involved. Subsequently, we provide necessary and sufficient conditions for the transition between states under deterministic covariant influences and necessary conditions in the presence of stochastic covariant influences, predicting which transitions between states are forbidden. Our approach, for the first time, employs the framework of resource theories to the study of finite discrete dynamical systems. The language of category theory allows us to articulate laws that unify the behaviour of different types of finite discrete dynamical systems, and their mathematical flavour makes them rigorous and checkable.

This extended abstract is based on [18].

Dynamical systems describe the evolution of several interesting situations. In many cases, when the evolution is particularly complex to deal with, one splits it into two parts: the uninfluenced part and the perturbation, where the latter is interpreted as a small correction to the uninfluenced evolution [1, 10]. In this work, we go beyond the perturbation paradigm by introducing the notion of *covariant influence*, which need not be a small contribution. The covariance condition guarantees that, despite not being small, it preserves the underlying structure representing the evolution of states. In this way, the evolution of a discrete dynamical system can be written as the composition of two evolutions: one is understood as the basic evolution of the system, and the other is the covariant influence. The covariance requirement ensures that the order in which these two parts are applied does not matter.

In order to develop a general theory of covariant influences in discrete dynamical systems with a finite number of states, we employ the framework of resource theories [4, 16, 3], grounded in category theory [6, 11, 13, 5]. This constitutes the first application of resource theories outside the physics domain, to a field with countless applications to diverse areas of science, including genetic regulatory networks [2, 9, 12].

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The general theory of covariant influences we develop comes in two flavours, corresponding to two different underlying process theories [8, 7] on which the resource theory is built. The first is a deterministic one, based on the process theory of sets and functions, where randomness is completely forbidden both in the initial state and in the action of the covariant influence. In the second, based on the process theory of stochastic maps, instead, we allow the presence of randomness both in the initial state and in the covariant influence. More precisely, the covariance condition is formulated as follows.

Definition 1. Let (A_1, ϕ_1) and (A_2, ϕ_2) be dynamical systems, where ϕ_i denotes the generator of the uninfluenced evolution of A_i. A map $f : A_1 \to A_2$ is called *covariant* if $f \circ \phi_1 = \phi_2 \circ f$.

Here A denotes a system type of the process theory: a set in the deterministic case, the simplex of probability vectors in the random case. We prove that the covariant condition on the processes of the corresponding process theories gives rise to a partitioned process theory in the sense of Ref. [6], i.e. that covariant maps form a strict symmetric monoidal subcategory of the underlying process theory. Note that, unlike in the quantum resource theory of asymmetry [15, 14, 17], where a similar definition is given, ϕ_i generates a monoid and not a group, i.e. ϕ_i is not invertible.

In such a setting, we analyze the issue of which transitions between the states of a dynamical system are possible under covariant influences. We show that deterministic influences allow hopping between attractors whose length becomes smaller and smaller, according to a divisibility criterion. Instead, in the presence of randomness, all jumps between attractors become possible and the divisibility criterion no longer holds. In particular, we achieve a full characterization of transitions between states in the deterministic setting through a complete family of resource monotones, and in the random case, we predict which transitions between states are forbidden.

To quote just the result for the deterministic case, in the process theories of sets and functions, we have three different monotones:

1. **Length** ℓ: the period of the attractor in the same basin of attraction as the element s;
2. **Transient progeny** d: the number of time steps necessary to go from the element s to its attractor;
3. **Ancestry** $a(s)$: the number of time steps necessary to reach the element s from its farthest predecessor.

Such quantities can be used to build a complete family of monotones for the resource theory.

Theorem 1. *Let s and s' be two deterministic states of a discrete dynamical system (S, ϕ). Then there exists a covariant influence converting s into s' if and only if*

$$d' \leq d, \tag{1}$$
$$\ell' \mid \ell, \tag{2}$$
$$a\left(\phi^n\left(s'\right)\right) \geq a\left(\phi^n\left(s\right)\right) \tag{3}$$

for $n = 0, \ldots, d' - 1$.

In conclusion, the use of resource theories and category theory allows us to get a unified picture of discrete dynamical systems under influences, regardless of the specifics of their evolutions, unlike most standard approaches to discrete dynamical systems where a concrete model needs to be postulated. In this way, our results, in the form of simple mathematical laws, can be phrased in general terms, so they are applicable to a broad class of discrete dynamical systems. The key concept in our analysis is covariance, which can be thought of as a symmetry in time evolution, and can be expressed by a simple commutativity condition.

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