The Algebra for Stabilizer Codes *(Extended Abstract)*

Full version: arXiv:2304.10584

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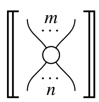
July 5, 2023

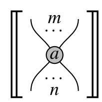
In this paper, we show that affine coisotropic relations yield a categorical semantics for stabilizer codes.

Background and motivation (reviewing material in the following publications [1, 2, 3])

The props of **linear relations** and **affine relations** have morphisms linear and affine subspaces. They are presented by interacting Hopf/Frobenius algebras. The white Frobenius algebra is induced by the copying comonoid and the grey Frobenius algebra by the addition monoid. In the prop of affine relations, the grey Frobenius algebra is additionally decorated by an element of the field corresponding to the affine shift. Importantly for the purposes of this paper, affine relations over $\mathbb{F}_p$ are isomorphic (modulo nonzero scalars) to the p-dimensional qudit phase free ZX-calculus in addition to the Pauli X gate:

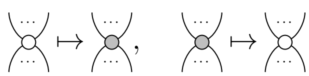


$$\propto \sum_{i=0}^{p-1} |i,\dots,i\rangle\langle i,\dots,i|$$



$$\propto \sum_{a+\sum x_i = \sum y_j \mod p} |y_1,\dots,y_m\rangle\langle x_1,\dots,x_n|$$

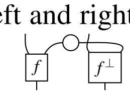
Also important for us, the colour swapping is the orthogonal complement in linear relations:

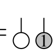


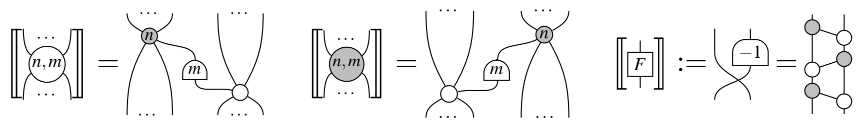
$$V \mapsto V^\perp := \{v \in \mathbb{F}_p^n : \forall w \in V, \langle v,w\rangle = 0\}$$

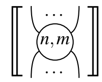
We can capture more of quantum theory using this relational semantics via linear symplectic geometry.

An even dimensional vector space $\mathbb{F}_p^{2n}$ is regarded as a **symplectic spector space** when it is equipped with a **symplectic form** $\omega : \mathbb{F}_p^{2n} \oplus \mathbb{F}_p^{2n} \to \mathbb{F}_p; ((z_0,x_0),(z_1,x_1)) \mapsto \langle z_0,x_1\rangle - \langle x_0,z_1\rangle$. Think of the vector space as graded into n position and a n momentum variables. A **Lagrangian subspace** of a symplectic vector space $(\mathbb{F}_p^{2n},\omega)$ is a linear subspace $V \subseteq \mathbb{F}_p^{2n}$ which is equal to its symplectic dual: $V = V^\omega := \{v \in \mathbb{F}_p^{2n} | \forall w \in V : \omega(v,w) = 0\}$. Lagrangian subspaces capture the maximally constrained linear evolution of position and momentum, asserting the commutation relations hold.

There prop of **Lagrangian relations**, $\mathsf{LagRel}_{\mathbb{F}_p}$, is the monoidal subcategory of linear relations whose maps are Lagrangian subspaces. *Incredibly*, $\mathsf{CPM}(\mathsf{LinRel}_{\mathbb{F}_p},\perp) \cong \mathsf{LagRel}_{\mathbb{F}_p}$, so that any Lagrangian relation can be factored into the following form; the left and right parts are colour swapped versions of each other, traced together by some number of caps: 

Adding the affine shift $|1\rangle :=$  to $\mathsf{LagRel}_{\mathbb{F}_p}$ we get the prop of **affine Lagrangian relations**, $\mathsf{AffLagRel}_{\mathbb{F}_p}$. This is isomorphic to **odd prime dimensional qudit stabilizer** modulo scalars, Stab_p (for $p = 2$ this is **Spekkens toy model**). A homogenized basis for the affine Lagrangian subspace corresponds to the stabilizer tableau for the corresponding stabilizer circuit. Alternatively, in ZX-calculus flavour, $\mathsf{AffLagRel}_{\mathbb{F}_p}$ is presented by two spiders both decorated by the group $(\mathbb{Z}/p\mathbb{Z})^2$ (the Fourier transform is derived):

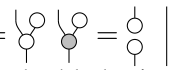
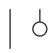


Interpreted as a complex matrix:  $\propto \sum_{j=0}^{p-1} e^{\pi \cdot i/p(n\cdot j + m\cdot j^2)} |j,\ldots,j\rangle\langle j,\ldots,j|$

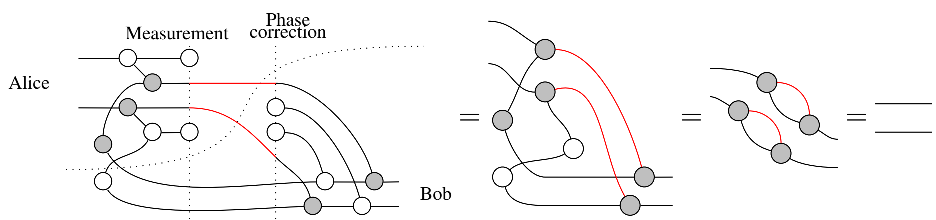
Adding measurement

We extend previous work by adding measurements to the picture. First we add quantum discarding; ie we give generators for $\mathsf{CPM}(\mathsf{Stab}_p) \cong \mathsf{CPM}(\mathsf{AffLagRel}_{\mathbb{F}_p})$ with respect to complex conjugation. That is, we give generators for **stabilizer codes** in this formalism.

This category can be captured by interpreting quantum discarding as the discarding *relation* ⚲⚲. Adding this to $\mathsf{AffLagRel}_{\mathbb{F}_p}$ yields the prop of **affine coisotropic relations**, $\mathsf{AffCoIsotRel}_{\mathbb{F}_p}$. This is defined in the same way as affine Lagrangian relations, except that the linear part merely *contains* its symplectic dual, so that $V^\omega \subseteq V$. This gives a larger degree of freedom of configurations of position and momentum. As a corollary, we get the incredible fact that $\mathsf{CPM}(\mathsf{CPM}(\mathsf{LinRel}_{\mathbb{F}_p})) \cong \mathsf{CoIsotRel}_{\mathbb{F}_p} \hookrightarrow \mathsf{CPM}(\mathsf{Stab}_p)$.

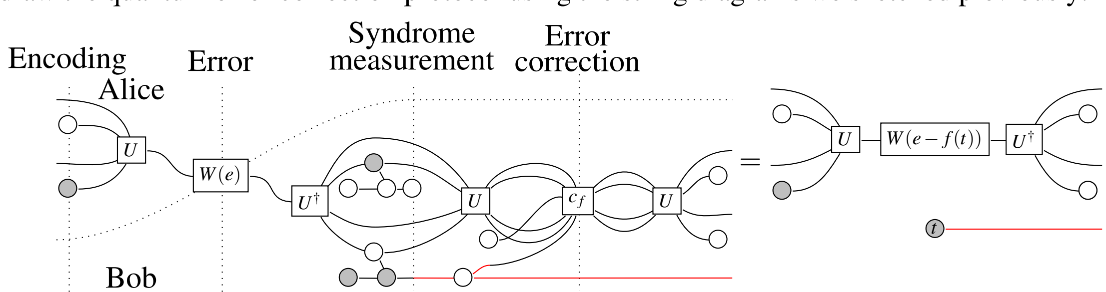
By splitting the Z projector $p_Z :=$ , we obtain a graphical calculus for mixed stabilizer circuits with two types of objects, doubled wires for quantum channels and single wires for classical channels. Concretely this is obtainted by adding two relations corresponding to state preparation and measurement in the Z basis:  : $\mathsf{Dit} \to \mathsf{Qudit}$ and ⚲ : $\mathsf{Qudit} \to \mathsf{Dit}$

For example, in this graphical calculus we can prove the correctness of the qudit quantum teleportation algorithm using only spider fusion (the classical wires are drawn in red for clarity):



Stabilizer codes and error correction protocols:

Take affine coisotropic $S = L + a \subseteq \mathbb{F}_p^{2n}$ where $L \subseteq \mathbb{F}_p^{2n}$ has dimension $n+k$. This is an $[n,k]$-stabilizer code, encoding k logical qudits in n physicals qudits. Fix an affine symplectic purification U of S. We can draw the quantum error correction protocol using the string diagrams we sketched previously:



Given any $(e_z, e_x) \in \mathbb{F}_p^{2n}$, interpret $W(e) := \bigotimes_{i=0}^{n-1} Z^{e_{z,i}} X^{e_{x,i}}$ as a Pauli bitstring error. The measurement $t \in \mathbb{F}_p^{n-k}$ is the **syndrome**. An error $W(e)$ is **undetectable** (the syndrome is zero) when $e \in L^\omega + a$. The map c_f is the classically controlled affine transformation $f : \mathbb{F}_p^{n-k} \to \mathbb{F}_p^{2n}$, chosen to correct the error.

References

[1] Filippo Bonchi, Robin Piedeleu, Paweł Sobociński & Fabio Zanasi (2019): *Graphical affine algebra*. In: *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, IEEE, pp. 1–12, doi:10.1109/LICS.2019.8785877. Available at http://www.zanasi.com/fabio/files/paperLICS19.pdf.

[2] Filippo Bonchi, Paweł Sobociński & Fabio Zanasi (2017): *Interacting Hopf algebras*. *Journal of Pure and Applied Algebra* 221(1), pp. 144–184, doi:10.1016/j.jpaa.2016.06.002. Available at https://arxiv.org/pdf/1403.7048.pdf.

[3] Cole Comfort & Aleks Kissinger (2022): *A Graphical Calculus for Lagrangian Relations*, doi:10.4204/eptcs.372.24. Available at https://doi.org/10.4204/eptcs.372.24.