Topological Quantum Gates in Homotopy Type Theory

David Jaz Myers[*] Hisham Sati[*,†] Urs Schreiber[*]

[*] Mathematics, Division of Science; and Center for Quantum and Topological Systems, NYUAD Research Institute, New York University Abu Dhabi, UAE.

[†]The Courant Institute for Mathematical Sciences, NYU, NY

The authors acknowledge the support by *Tamkeen* under the *NYU Abu Dhabi Research Institute grant* `CG008`.

In this talk, we will propose a remarkably simple formalization of topological quantum gates in homotopy type theory as transport in a certain type family, following our paper [MSS23] of the same title. To understand how our formalization can be so simple — derivable from bare foundations in a matter of forty pages, a feat inconceivable in set theoretic foundations — we must first understand what a topological quantum gate is expected to be, and how a realistic class of such gates is describable in synthetic homotopy theory.

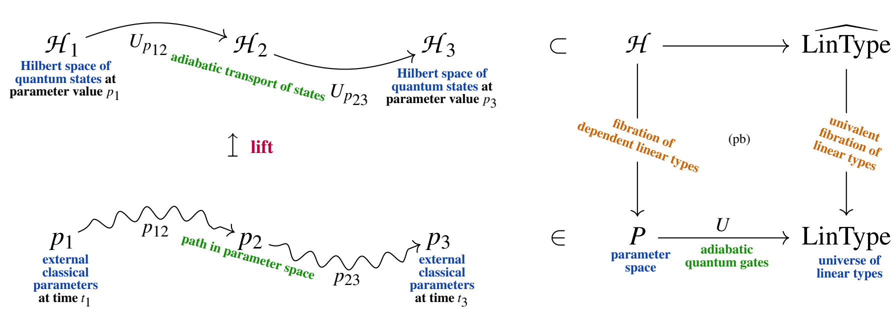
A quantum system is usually represented by its Hilbert space $\mathscr{H}$ of states, but these systems may also depend on certain classical environmental parameters such as the experimental setup or conditions of materials in use during an experiment. Therefore, rather than being represented by a single Hilbert space, a quantum system in practice is represented by a bundle $\mathscr{H} \to P$ of Hilbert spaces $\mathscr{H}_p$ over a classical parameter space P. Furthermore, we expect that continuous changes in the parameter p will induce changes in the state of the system; that is, we equip the bundle $\mathscr{H} \to P$ with a connection so that parallel transport along a path $p_{12} : p_1 \rightsquigarrow p_2$ yields a function $\mathscr{H}_{p_1} \to \mathscr{H}_{p_2}$. We can therefore drive our quantum systems by giving continuous changes in the classical parameters.

Quantum systems are very susceptible to noise from their environment, getting more susceptible to decoherence as the systems get larger. This provides an obstacle to developing quantum computers with many qbits, which has lead researchers to search for quantum systems which are *topologically insulated* from their environment. Roughly speaking, a quantum system $\mathscr{H} \to P$ is topologically insulated from its environment if the connection on this bundle is flat — that is, if the parallel transport of states over continuous changes of environment variables is homotopy invariant. A topologically insulated system will therefore be insulated from noisy perturbations to paths in environment variables which are sufficiently small as to be homotopic to the intended path.

A reversible logic gate is an automorphism of $\{0,1\}^n$ — a reversible transformation of the state of n bits. By analogy, a quantum gate is a unitary function of the state of n qbits, that is, a unitary automorphism of $\mathbb{C}^{2n}$. If a quantum gate is implemented by parallel transport in a topologically insulated system, we say that it is a topological quantum gate.

Topological quantum gates are expected to be implemented by "braiding anyonic defects" of the (topologically ordered) *ground states* of an essentially 2-dimensional topological quantum material. The potential for this is neatly captured by one of the classical theorems of mathematical quantum mechanics: the *Quantum Adiabatic Theorem*, which says that in the asymptotic limit of sufficiently gentle (i.e. "adiabatic") changes in external classical parameters, the induced evolution of the system asymptotically preserves gapped energy eigen-states, and so in particular preserves gapped ground states. Such adiabatic changes therefore act on the Hilbert spaces $\mathcal{H}_{p_1}$ of gapped ground states by unitary operators $U_{p_{12}}$ that vary continuously with the adiabatic parameter path $p_{12} : p_1 \rightsquigarrow p_2$.

Figure T. Schematically shown on the left is the "adiabatic" transport of quantum states along linear maps depending on continuous paths in a classical parameter space.
The diagram on the right indicates our description of such situations by (linear) homotopy type families depending on a base homotopy type, where we interpret the variation of states over paths as an instance of the *transport rule* of homotopy type theory.



These topological quantum materials are expected to be essentially 2-dimensional lattices with point "anyonic" defects. For this reason, the classical parameter space P of such a system is expected to be the configuration space $\mathbf{B}PBr(N)$ of N points on a plane. The Hilbert spaces $\mathscr{H}_p$ depending on the position $p \in \mathbf{B}PBr(N)$ of the anyonic defects are thought to be the spaces of states of a Chern-Simons field theory, or rather their "chiral half", called the spaces of "conformal blocks" (see [FMS97, §C]). A derivation of this (previously unproven, cf. [Va21]) assumption from first principles is argued in [SS22-Ord, §3].

This bundle of conformal blocks carries a canonical flat connection known as the *Knizhnik-Zamolodchikov connection*, [FMS97, §15.3.2][Ko02, §1.5, 2.1][Ab15, §4], which is meant to give the (topologically insulated) evolution of the system. The parallel transport of this KZ-connection constitutes a unitary representation of the fundamental group of the configuration space of points, and so a representation of the pure braid group. This is the sense in which a topological quantum gate may be implemented by braiding anyonic defects.

Remarkably, the *hypergeometric integral construction* ([SS22-Def, Prop. 2.15, 2.17]) in conformal field theory provides an equivalence between

(1) the bundle of conformal blocks with its KZ-connection

(2) the bundle of suitably twisted cohomology groups of configuration spaces of $N+\bullet$ points equipped with their (flat) Gauss-Manin connection.

The Gauss-Manin connection is a flat connection definable on the twisted cohomology groups of a bundle using purely homotopical methods. It is this remarkable equivalence which allows us to identify these bundles of conformal blocks (representing topological quantum systems) with constructions in abstract homotopy theory that may be formalized in homotopy type theory.

We will understand parallel transport in the Gauss-Manin connection as transport in a type family of twisted cohomology groups. We will then construct the configuration spaces $\mathbf{B}PBr(N)$ in homotopy type theory using a novel construction, and define the appropriate twisted family of cohomology groups. Thanks to the *hypergeometric integral construction*, this will equivalently provide a formalization of the KZ-connection on conformal blocks, which we conclude as our main Theorem. The formalization in Cubical Agda of this approach is ongoing as part of an undergraduate research program at CQTS.

Definition 1 (Gauss-Manin transport on fibrations of twisted cohomology groups)**.** We say that the data type of *fibrations of twisted ordinary cohomology sets* is:

$$\left.\begin{array}{l} \overset{\text{coefficients}}{R : \mathrm{Ring}},\ \overset{\text{degree}}{n : \mathbb{N}},\ \overset{\text{parameter base}}{B : \mathrm{Type}}, \\ \underset{\text{fibration of domains}}{X_{(-)} : B \to \mathrm{Type}},\ \underset{\text{family of twists}}{\tau_{(-)} : (b : B) \to \big(X_b \to \mathbf{B}R^\times\big)} \end{array}\right\} \vdash \begin{array}{l} \overset{\text{twisted cohomology}}{H^{n+\tau_{(-)}}\big(X_{(-)};R\big)} :\equiv \\ \Big[(t : \mathbf{B}R^\times) \to \Big(\underbrace{\mathrm{fib}_t\big(\tau_{(-)}\big)}_{\mathrm{fib}_{(-,t)}(\mathrm{pr}_X,\tau)} \to \mathbf{B}^n\big(\circlearrowright_t R_{\mathrm{udl}}\big)\Big)\Big]_0 \end{array} \ : \underset{\text{over base}}{\overset{\text{fibered}}{B \to \mathrm{Type}}} \tag{1}$$

Given such, its *Gauss-Manin parallel transport* is the corresponding transport over the base type B:

$$\begin{array}{rl} \mathrm{GMTransport} : \prod\limits_{b_1,b_2:B} \Big(\mathrm{Id}_B(b_1,b_2) & \longrightarrow \Big(H^{n+\tau_{b_1}}\big(X_{b_1};R\big) \xrightarrow{\sim} H^{n+\tau_{b_2}}\big(X_{b_2};R\big)\Big)\Big). \\ \big(b_1 \overset{p_{12}}{\rightsquigarrow} b_2\big) & \longmapsto \qquad (p_{12})_* \end{array} \tag{2}$$

Definition 2 (Homotopy data structure of conformal blocks)**.** In specialization of Def. 1, we obtain this data type:

$$\left.\begin{array}{l} \overset{\text{punctures}}{N : \mathbb{N}_+},\ \overset{\text{degree}}{n : \mathbb{N}},\ \overset{\text{shifted level}}{\kappa : \mathbb{N}_{\geq 2}} \\ \underset{\text{weights}}{\mathrm{w}_{(-)} : N \to \{0,\cdots,\kappa-2\}} \end{array}\right\} \vdash \Big(\vec{z} \mapsto \Big[(t : \mathbf{B}\mathbb{C}^\times) \to \Big(\mathrm{fib}_{(t,\vec{z})}\big(\mathrm{pr}^{N+n}_N,\ \tau_{(\kappa,\mathrm{w}_\bullet)}\big) \to \mathbf{B}^n\big(\circlearrowright_t \mathbb{C}_{\mathrm{udl}}\big)\Big)\Big]_0\Big) : \mathbf{B}\,\mathrm{PBr}(N) \to \mathrm{Type}$$

where

$$\begin{array}{rccc} \mathrm{pr}^{N+n}_N : & \mathbf{B}\,\mathrm{PBr}(N+n) & \longrightarrow & \mathbf{B}\,\mathrm{PBr}(N) \\ & \mathrm{pt}\,\underset{b_{Ii}}{\circlearrowright} & \longmapsto & \mathrm{pt}\,\underset{\mathrm{e}}{\circlearrowright} \\ & \mathrm{pt}\,\underset{b_{IJ}}{\circlearrowright} & \longmapsto & \mathrm{pt}\,\underset{b_{IJ}}{\circlearrowright} \end{array} \qquad \begin{array}{rccc} \tau_{(\kappa,\mathrm{w}_\bullet)} : & \mathbf{B}\,\mathrm{PBr}(N+n) & \longrightarrow & \mathbf{B}\,\mathbb{C}^\times \\ & \mathrm{pt}\,\underset{b_{Ii}}{\circlearrowright} & \longmapsto & \mathrm{pt}\,\underset{\exp(2\pi\mathrm{i}\frac{\mathrm{w}_I}{\kappa})}{\circlearrowright} \\ & \mathrm{pt}\,\underset{b_{ij}}{\circlearrowright} & \longmapsto & \mathrm{pt}\,\underset{\exp(2\pi\mathrm{i}\frac{2}{\kappa})}{\circlearrowright} \\ & \mathrm{pt}\,\underset{b_{IJ}}{\circlearrowright} & \longmapsto & \mathrm{pt}\,\underset{\exp(2\pi\mathrm{i}\frac{\mathrm{w}_I\mathrm{w}_J}{2\kappa})}{\circlearrowright} \end{array} \tag{3}$$

Theorem 3 (**Topological quantum gates as homotopy data structure**)**.** *The semantics in the classical model topos of the transport operation* (2) *in this data type* (3) *is given by the monodromy of the Knizhnik-Zamolodchikov connection, on* $\widehat{\mathfrak{su}_2}^{\kappa-2}$*-conformal blocks (on the Riemann sphere with* $N+1$ *punctures weighted by* $(\mathrm{w}_I)_{I=1}^N$ *and* $\mathrm{w}_{N+1} = n + \sum_I \mathrm{w}_I$*).*

References

[Ab15] C. A. Abad, *Introduction to representations of braid groups*, Rev. Colomb. Mat. **49** 1 (2015) 1, [arXiv:1404.0724], [doi:10.15446/recolma.v49n1.54160].

[DJMM90] E. Date, M. Jimbo, A. Matsuo, and T. Miwa, *Hypergeometric-type integrals and the* $\mathfrak{sl}(2,\mathbb{C})$ *Knizhnik-Zamolodchikov equation*, Int. J. Mod. Phys. B **4** 5 (1990), 1049-1057, [doi:10.1142/S0217979290000528].

[EFK98] P. I. Etingof, I. Frenkel, and A. A Kirillov, *Lectures on Representation Theory and Knizhnik-Zamolodchikov Equations*, Math. Surv. monogr. **58**, Amer. Math. Soc., Providence, RI (1998), [ams.org/surv-58].

[FSV94] B. Feigin, V. Schechtman, A. Varchenko, *On algebraic equations satisfied by hypergeometric correlators in WZW models. I*, Commun. Math. Phys. **163** (1994), 173–184, [doi:10.1007/BF02101739].

[FMS97] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal field theory*, Springer, Berlin (1997), [doi:10.1007/978-1-4612-2256-9].

[Ko02] T. Kohno, *Conformal field theory and topology*, Transl Math. Monogr. **210**, Amer. Math. Soc., Providence, RI (2002), [ams:mmono-210].

[MSS23] David Jaz Myers, H. Sati, and U. Schreiber, *Topological Quantum Gates in Homotopy Type Theory*, arXiv preprint (2023),
[arXiv:2303.02382].

[SS22-Def] H. Sati and U. Schreiber, *Anyonic defect branes in TED-K-theory*, Rev. Math. Phys. (2023), [doi:10.1142/S0129055X23500095], [arXiv:2203.11838].

[SS22-Ord] H. Sati and U. Schreiber, *Anyonic topological order in TED-K-theory*, Rev. Math. Phys. **35** 03 (2023) 2350001, [doi:10.1142/S0129055X23500010], [arXiv:2206.13563].

[SS22-TQC] H. Sati and U. Schreiber, *Topological Quantum Programming in* TED-K, PlanQC **2022** 33 (2022), [arXiv:2209.08331], [ncatlab.org/schreiber/show/Topological+Quantum+Programming+in+TED-K].

[Va21] S. J. Valera, *Fusion Structure from Exchange Symmetry in* $(2+1)$*-Dimensions*, Ann. Phys. **429** (2021) 168471, [doi:10.1016/j.aop.2021.168471], [arXiv:2004.06282].

[ZR99] P. Zanardi and M. Rasetti, *Holonomic Quantum Computation*, Phys. Lett. A **264** (1999), 94-99, [doi:10.1016/S0375-9601(99)00803-8], [arXiv:quant-ph/9904011].

[QP] *Quantum Programming via Linear Homotopy Types*, in preparation.