Structured and decorated cospans from the viewpoint of double category theory

Evan Patterson
Topos Institute
evan@epatters.org

Structured and decorated cospans are broadly applicable frameworks for building bicategories or double categories of open systems. We streamline and generalize these frameworks using central concepts of double category theory. We show that, under mild hypotheses, double categories of structured cospans are cocartesian (have finite double-categorical coproducts) and are equipments. The proofs are simple as they utilize appropriate double-categorical universal properties. Maps between double categories of structured cospans are studied from the same perspective. We then give a new construction of the double category of decorated cospans using the recently introduced double Grothendieck construction. Besides its conceptual value, this reconstruction leads to a natural generalization of decorated cospans, which we illustrate through an example motivated by statistical theories and other theories of processes.

1 Introduction

A central theme of applied category theory is the mathematical modeling of open systems: physical or computational systems that interact with each other along boundaries or interfaces. Within this tradition, mathematical models of open systems are most commonly based on spans or cospans, an idea now at least twenty-five years old [17, 22]. Two general frameworks for building open systems using cospans have emerged: structured cospans [5, 11] and decorated cospans [6, 12]. Complementing the mathematical theory, structured cospans have been implemented in the programming framework Catlab.jl and used to create software tools for epidemiological modeling based on open Petri nets [7, 18] and open stock and flow diagrams [3]. Structured and decorated cospans are now essential tools of applied and computational category theory.

The categorical description of open systems based on cospans has evolved over time. Some early works studied categories of cospans, which compose by taking pushouts. Because pushouts are defined only up to isomorphism, the morphisms of these categories must be *isomorphism classes* of cospans. This is unfaithful to implementation, where one always computes with representatives of an equivalence class, rather than the equivalence class itself. More fundamentally, systems generally have morphisms of their own—for example, Petri nets come with homomorphisms between them—and these are lost if open systems are taken to be morphisms, rather than objects, of a category.

Both problems are solved by passing from categories to a two-dimensional categorical structure, of which the best-studied are bicategories. Yet this presents its own difficulties. In addition to composing along their boundaries, open systems generally admit a symmetric monoidal product that juxtaposes two of them "in parallel." One then needs to construct not just a bicategory but a symmetric monoidal bicategory of open systems. Monoidal bicategories are inherently complicated because they are properly a three-dimensional categorical structure (namely, tricategories with one object). It was noticed that rather than constructing a monoidal bicategory directly, it can be easier to first construct a monoidal *double* *category* and then obtain the monoidal bicategory from the globular cells of the double category [16, 24]. But since a double category is at least as good as a bicategory, one may as well consider double categories of open systems. That is now what is typically done. In recent work, both structured cospans [5] and decorated cospans [6] have been assembled into symmetric monoidal double categories.

The thesis of this paper is that viewing open systems as double categories is not merely a technical device or a means to constructing bicategories, but a source of mathematical insights that cannot be obtained at the 1-categorical or even bicategorical levels. To understand why, consider the philosophy behind the modern theory of double categories, as developed principally by Grandis and Paré, beginning with an account of double limits and colimits [15], and exposited recently by Grandis [14]. Another important expression of this viewpoint is Shulman's theory of equipments [23].

A **double category** is succinctly defined as a pseudocategory in **Cat**.[1] Thus, a double category $\mathbb{D}$ consists of a category of objects, $\mathbb{D}_0$; a category of morphisms, $\mathbb{D}_1$; source and target functors, $\mathrm{src}, \mathrm{tgt} : \mathbb{D}_1 \rightrightarrows \mathbb{D}_0$; and external composition and identity operations, $\odot : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$ and $\mathrm{id} : \mathbb{D}_0 \to \mathbb{D}_1$, which obey the category axioms up to coherent globular isomorphisms in $\mathbb{D}_1$. The objects of the category $\mathbb{D}_0$ are called the **objects** of the double category $\mathbb{D}$, the morphisms of $\mathbb{D}_0$ the **arrows** of $\mathbb{D}$, the objects of $\mathbb{D}_1$ the **proarrows** of $\mathbb{D}$, and the morphisms of $\mathbb{D}_1$ the **cells** of $\mathbb{D}$. In particular, on this definition, the proarrows of a double category are first and foremost the *objects* of a category, which happen to have a source and target. In important examples of double categories, such as those of spans, cospans, relations, matrices, profunctors, and bimodules, the proarrows are best thought of in precisely this way, as objects that happen to have a source and target. Crucially, this also applies to double categories of open systems, which are systems that happen to have boundaries. Shulman calls such double categories (or rather their underlying bicategories) "$\mathscr{M}od$-like" after the bicategory of bimodules between rings [23].

Whether one thinks of proarrows primarily as objects or morphisms may seem a small matter of perspective, but it gains significance through the modern theory of double categories, where proarrows play the role of objects in all of the main concepts, such as natural transformations, limits and colimits, commas, adjunctions, and the Grothendieck construction. The theory is thus well suited to describe open systems, including those based on spans and cospans. In this paper, we study structured and decorated cospans from the viewpoint of double category theory.[2]

After reviewing their structure as a double category, we show that structured cospans form a cocartesian double category, a statement that is stronger yet easier to prove than being a symmetric monoidal double category. We also show that structured cospans are an equipment, so altogether form a cocartesian equipment (Section 2). Here we see the advantages of double-categorical universal properties. We then turn to decorated cospans (Section 3), reconstructing the double category of decorated cospans as an application of the double Grothendieck construction [10]. As a byproduct, we also generalize decorated cospans in several directions, which we illustrate through an example motivated by categorical statistics.

2 Structured cospans as a cocartesian equipment

Structured cospans represent open systems as cospans whose feet are restricted compared with the apex [5, 11]. A simple example is open graphs with boundaries restricted to be *discrete* graphs. Compared with other techniques, structured cospans have the advantage of being particularly easy to use, as the

[1] Some authors call this structure a **pseudo double category** but since all double categories in this paper are pseudo, we prefer to omit the adjective. Likewise, our double functors are pseudo by default. For complete definitions of these concepts, see [14].

[2] This paper synthesizes a series of blog posts by the author: "Grothendieck construction for double categories" (2022), "Decorated cospans via the Grothendieck construction" (2022), "Structured cospans as a cocartesian equipment" (2023).

hypotheses for the construction are often easy to check in examples. However, proofs that the construction itself works are more involved because the mathematical object being constructed—a symmetric monoidal double category—is complicated, involving a large number of coherence conditions. Three different correctness proofs for structured cospans have been given: the first one by direct but lengthy verification of the axioms [9] and two later ones by more conceptual routes that however import other sophisticated concepts. These concepts are pseudocategories in the 2-category of symmetric monoidal categories [5] and symmetric monoidal bifibrations [6].

Such difficulties can be bypassed by viewing structured cospans in a different light, as forming a *cocartesian* double category, even a cocartesian equipment. Just as a cartesian or cocartesian category can be given the structure of a symmetric monoidal category by making a choice of finite products or coproducts, so can a cartesian or cocartesian double category be given the structure of a symmetric monoidal double category. It is, however, much easier to prove cocartesianness than to directly construct the symmetric monoidal product. This circumstance highlights a recurring tension in category theory: that between universal properties and algebraic structures. Although algebraic structure is arguably more flexible, universal properties, when they can be found, are extremely powerful because many consequences and coherences flow directly from the defining existence and uniqueness statement, which is often easy to verify in particular situations. Both cocartesian double categories and equipments are defined by universal properties, whereas a symmetric monoidal product is a structure on a double category.

2.1 Double category of structured cospans

We begin by reviewing the definition of structured cospans and their structure as a double category [5].

Proposition 2.1. *Let $L : \mathsf{A} \to \mathsf{X}$ be a functor into a category X with pushouts. Then there is a double category ${}_L\mathbb{C}\mathsf{sp}(\mathsf{X})$ that has*

- *as objects, the objects of A;*
- *as arrows, the morphisms of A;*
- *as proarrows $a \nrightarrow b$, **L-structured cospans** with feet a and b, which are cospans in X of the form $La \to x \leftarrow Lb$;*
- *as cells $\begin{array}{ccc} a & \overset{x}{\nrightarrow} & b \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ c & \underset{y}{\nrightarrow} & d \end{array}$, **morphisms of L-structured cospans** with foot maps f and g, which are morphisms of cospans in X of the form*

$$\begin{array}{ccccc} L(a) & \longrightarrow & x & \longleftarrow & L(b) \\ {\scriptstyle Lf}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Lg} \\ L(d) & \longrightarrow & y & \longleftarrow & L(c) \end{array}.$$

Composition in the categories ${}_L\mathbb{C}\mathsf{sp}(\mathsf{X})_0$ and ${}_L\mathbb{C}\mathsf{sp}(\mathsf{X})_1$ is by composition in A and $\mathbb{C}\mathsf{sp}(\mathsf{X})_1$, respectively, and external composition in ${}_L\mathbb{C}\mathsf{sp}(\mathsf{X})$ is given by that in $\mathbb{C}\mathsf{sp}(\mathsf{X})$, i.e., by pushout in X.

We take this result as given. The proof is straightforward because the double category structure of L-structured cospans is inherited from that of cospans in X. For details, see [5, Theorem 2.3].

2.2 Cocartesian equipment of structured cospans

We now prove that structured cospans form an equipment, then a cocartesian double category, and hence a cocartesian equipment. The reader may find it helpful to review the definitions of cocartesian double categories and equipments in Appendix A.

Proposition 2.2. *Let $L : \mathsf{A} \to \mathsf{X}$ be a functor into a category X with pushouts. Then the double category of L-structured cospans is an equipment.*

Proof. To restrict an L-structured cospan $(c, Lc \to y \leftarrow Ld, d)$ along arrows $f : a \to c$ and $g : b \to d$ in A, simply restrict the underlying cospan in X along Lf and Lg, using the fact that $\mathbb{C}\mathsf{sp}(\mathsf{X})$ is an equipment (Example A.3). The universal property holds as a special case of the universal property in $\mathbb{C}\mathsf{sp}(\mathsf{X})$:

$$
\begin{array}{ccccc}
La' & \longrightarrow & x & \longleftarrow & Lb' \\
{\scriptstyle Lf'}\downarrow & & \Big| & & \downarrow{\scriptstyle Lg'} \\
La & & {\scriptstyle h} & & Lb \\
{\scriptstyle Lf}\downarrow & & \downarrow & & \downarrow{\scriptstyle Lg} \\
Lc & \underset{\ell}{\longrightarrow} & y & \underset{r}{\longleftarrow} & Ld
\end{array}
\quad = \quad
\begin{array}{ccccc}
La' & \longrightarrow & x & \longleftarrow & Lb' \\
{\scriptstyle Lf'}\downarrow & & {\scriptstyle h}\,\vdots & & \downarrow{\scriptstyle Lg'} \\
La & \underset{\ell\circ Lf}{\longrightarrow} & y & \underset{r\circ Lg}{\longleftarrow} & Lb \\
{\scriptstyle Lf}\downarrow & & \| & & \downarrow{\scriptstyle Lg} \\
Lc & \underset{\ell}{\longrightarrow} & y & \underset{r}{\longleftarrow} & Ld
\end{array}
$$

□

For the double category of L-structured cospans to be cocartesian, extra assumptions are needed. Clearly, the category A must itself have finite coproducts. Also, these must preserved by the functor $L : \mathsf{A} \to \mathsf{X}$. The latter is often is easy to verify in examples by exhibiting L as a left adjoint.

Theorem 2.3. *Suppose A is a category with finite coproducts, X is a category with finite colimits, and $L : \mathsf{A} \to \mathsf{X}$ is a functor that preserves finite coproducts. Then the double category of L-structured cospans is cocartesian, hence also a cocartesian equipment.*

Proof. Because the categories A and X have finite coproducts, there are canonical comparison maps

$$L_{a,a'} := [L(\iota_a), L(\iota_{a'})] : L(a) + L(a') \to L(a + a'), \qquad a, a' \in \mathsf{A},$$

and $L_0 := !_{L(0)} : 0_{\mathsf{X}} \to L(0_{\mathsf{A}})$. For any maps $f : a \to c$ and $f' : a' \to c$ in A, the comparisons satisfy

$$
\begin{array}{ccc}
L(a) + L(a') & \xrightarrow{\quad L_{a,a'} \quad} & L(a+a') \\
& {\scriptstyle [Lf, Lf']}\searrow \quad \swarrow{\scriptstyle L([f,f'])} & \\
& L(c) &
\end{array}
$$

as shown by precomposing both sides with the coprojections ι_{La} and $\iota_{La'}$ to obtain Lf and Lf', respectively. Since by assumption L preserves finite coproducts, the comparisons $L_{a,a'}$ and L_0 are, in fact, isomorphisms.

We now prove that the categories underlying ${}_L\mathbb{C}\mathsf{sp}(\mathsf{X})$ are cocartesian. By assumption, the category ${}_L\mathbb{C}\mathsf{sp}(\mathsf{X})_0 = \mathsf{A}$ has finite coproducts. Since the comparison L_0 is an isomorphism, $L(0_{\mathsf{A}})$ is initial in X and the initial L-structured cospan is $(0_{\mathsf{A}}, \mathrm{id}_{L(0_{\mathsf{A}})}, 0_{\mathsf{A}})$. Furthermore, the coproduct of two L-structured cospans $(a, La \to x \leftarrow Lb, b)$ and $(a', La' \to x' \leftarrow Lb', b')$, denoted

$$(a + a',\ L(a + a') \to x + x' \leftarrow L(b + b'),\ b + b'),$$

is obtained from the pointwise coproduct of cospans in X by restriction along the inverse comparisons $L^{-1}_{a,a'}$ and $L^{-1}_{b,b'}$. The universal property of coproducts in ${}_L\mathbb{C}\mathsf{sp}(\mathsf{X})_1$ then takes the form:

$$
\begin{array}{ccccccccc}
L(a+a') & \xrightarrow{L^{-1}_{a,a'}} & L(a) + L(a') & \longrightarrow & x + x' & \longleftarrow & L(b) + L(b') & \xleftarrow{L^{-1}_{b,b'}} & L(b+b') \\
{\scriptstyle L([f,f'])}\downarrow & & {\scriptstyle [Lf,Lf']}\downarrow & & {\scriptstyle [h,h']}\downarrow & & \downarrow{\scriptstyle [Lg,Lg']} & & \downarrow{\scriptstyle L([g,g'])} \\
L(c) & = & L(c) & \longrightarrow & y & \longleftarrow & L(d) & = & L(d)
\end{array}
.
$$

We have shown that both categories underlying ${}_L\mathbb{C}\mathsf{sp}(\mathsf{X})$ have finite coproducts, and it is immediate that the source and target functors preserve them.

Finally, the comparison cells in ${}_L\mathbb{C}\mathsf{sp}(\mathsf{X})$ interchanging finite coproducts with external composition and identity (Definition A.1) are all isomorphisms because they are defined by the same maps in X as the comparison cells in $\mathbb{C}\mathsf{sp}(\mathsf{X})$, which we already know to be isomorphisms (Example A.3). □

As a corollary, every double category of L-structured cospans satisfying the hypotheses of the theorem can be given the structure of a symmetric monoidal double category, by making choices of coproducts in both underlying categories. This follows abstractly because any cocartesian object in a 2-category with finite 2-products is a symmetric pseudomonoid in a canonical way [24, Remark 2.11]. Cocartesian double categories are cocartesian objects in the 2-category **Dbl**, whereas symmetric monoidal double categories are symmetric pseudomonoids in **Dbl**.

2.3 Maps between structured cospan double categories

We complete the essential theory of structured cospans by showing how to construct maps between cocartesian equipments of structured cospans. These maps are cocartesian double functors (Definition A.4). Compared with the original results [5, Theorems 4.2 and 4.3], the theorem below is slightly more general, treating the lax case as well as the pseudo one, and slightly stronger, yielding cocartesian double functors instead of symmetric monoidal ones.

Theorem 2.4. *Suppose we have a diagram in* **Cat** *of the form*

$$\begin{array}{ccc} \mathsf{A} & \xrightarrow{L} & \mathsf{X} \\ {\scriptstyle F_0}\big\downarrow & \overset{\alpha}{\nearrow} & \big\downarrow{\scriptstyle F_1} \\ \mathsf{A}' & \xrightarrow[L']{} & \mathsf{X}' \end{array},$$

where the categories X *and* X' *have pushouts. Then there is a lax double functor* $\mathbb{F} : {}_L\mathbb{C}\mathsf{sp}(\mathsf{X}) \to {}_{L'}\mathbb{C}\mathsf{sp}(\mathsf{X}')$ *that has underlying functor* $\mathbb{F}_0 = F_0$ *and acts on proarrows as*

$$(a, L(a) \xrightarrow{\ell} x \xleftarrow{r} L(b), b) \quad \mapsto \quad (F_0(a), L'(F_0(a)) \xrightarrow{\alpha_a} F_1(L(a)) \xrightarrow{F_1(\ell)} F_1(x) \xleftarrow{F_1(r)} F_1(L(b)) \xleftarrow{\alpha_b} L'(F_0(b)), F_0(b))$$

and on cells as

$$\begin{array}{ccccc} L(a) & \xrightarrow{\ell} & x & \xleftarrow{r} & L(b) \\ {\scriptstyle Lf}\big\downarrow & & {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle Lg} \\ L(a') & \xrightarrow[\ell']{} & x' & \xleftarrow[r']{} & L(b') \end{array} \quad \mapsto \quad \begin{array}{ccccc} L'(F_0(a)) & \xrightarrow{F_1(\ell)\circ\alpha_a} & F_1(x) & \xleftarrow{F_1(r)\circ\alpha_b} & L'(F_0(b)) \\ {\scriptstyle L'(F_0(f))}\big\downarrow & & {\scriptstyle F_1(h)}\big\downarrow & & \big\downarrow{\scriptstyle L'(F_0(g))} \\ L'(F_0(a')) & \xrightarrow[F_1(\ell')\circ\alpha_{a'}]{} & F_1(x') & \xleftarrow[F_1(r')\circ\alpha_{b'}]{} & L'(F_0(b')) \end{array}.$$

Moreover, $\mathbb{F}$ *is a pseudo double functor whenever* F_1 *preserves pushouts and* α *is a natural isomorphism.*

Suppose further that all of the categories in question have finite coproducts and that L *and* L' *preserve them, so that both double categories* ${}_L\mathbb{C}\mathsf{sp}(\mathsf{X})$ *and* ${}_{L'}\mathbb{C}\mathsf{sp}(\mathsf{X}')$ *are cocartesian. Then the lax double functor* $\mathbb{F}$ *is cocartesian if and only if both functors* F_0 *and* F_1 *are cocartesian. In particular,* $\mathbb{F}$ *is a cocartesian pseudo double functor whenever* F_0 *preserves finite coproducts,* F_1 *preserves finite colimits, and* α *is a natural isomorphism.*

As a substantial application of the theorem, we have formulated the generalized Lokta-Volterra model as a cocartesian lax double functor from open signed graphs to open parameterized dynamical systems [1].

We prove the theorem by decomposing the lax double functor $\mathbb{F}$ into three simpler ones. Taken together, the lemmas also implicitly give formulas for the laxators and unitors of $\mathbb{F}$, which we omitted in the theorem statement.

Lemma 2.5. *Let X be a category with pushouts and let $\mathsf{A}_0 \xrightarrow{F_0} \mathsf{A} \xrightarrow{L} \mathsf{X}$ be functors. Then there is a* strict *double functor ${}_{F_0}\mathbb{C}\mathsf{sp}(\mathsf{X}) : {}_{L\circ F_0}\mathbb{C}\mathsf{sp}(\mathsf{X}) \to {}_{L}\mathbb{C}\mathsf{sp}(\mathsf{X})$ given by F_0 on objects and arrows and by the identity on the cospans and maps of cospans underlying proarrows and cells.*

Furthermore, the double functor ${}_{F_0}\mathbb{C}\mathsf{sp}(\mathsf{X})$ is cocartesian whenever A_0, A, and X have finite coproducts and the functors L and F_0 preserve them.

The proof is immediate from the definitions. The next lemma is slightly more involved.

Lemma 2.6. *Let X and X' be categories with pushouts and let $\mathsf{A} \xrightarrow{L} \mathsf{X} \xrightarrow{F_1} \mathsf{X}'$ be functors. Then there is a* normal *lax double functor ${}_{L}\mathbb{C}\mathsf{sp}(F_1) : {}_{L}\mathbb{C}\mathsf{sp}(\mathsf{X}) \to {}_{F_1\circ L}\mathbb{C}\mathsf{sp}(\mathsf{X}')$ that is the identity on objects and arrows and acts on proarrows and cells by postcomposing the underlying diagrams in X with $F_1 : \mathsf{X} \to \mathsf{X}'$.*

The laxators are given by the universal property of pushouts in X', and ${}_{L}\mathbb{C}\mathsf{sp}(F_1)$ is pseudo if and only if F_1 preserves pushouts. Furthermore, when A, X, and X' have finite coproducts and L preserves them, ${}_{L}\mathbb{C}\mathsf{sp}(F_1)$ is cocartesian if and only if F_1 is cocartesian.

Proof. The proposed lax double functor ${}_{L}\mathbb{C}\mathsf{sp}(F_1) : {}_{L}\mathbb{C}\mathsf{sp}(\mathsf{X}) \to {}_{F_1\circ L}\mathbb{C}\mathsf{sp}(\mathsf{X}')$ acts on cospans and maps of cospans in exactly the same way as the lax double functor $\mathbb{C}\mathsf{sp}(F_1) : \mathbb{C}\mathsf{sp}(\mathsf{X}) \to \mathbb{C}\mathsf{sp}(\mathsf{X}')$ reviewed in Example A.5. The proof thus carries over directly. □

In the final lemma, we isolate the maps between structured cospan double categories induced by natural transformations between the structuring functors.

Lemma 2.7. *Let X be a category with pushouts and let $\alpha : L' \Rightarrow L : \mathsf{A} \to \mathsf{X}$ be a natural transformation. Then there is a lax double functor $\alpha^* : {}_{L}\mathbb{C}\mathsf{sp}(\mathsf{X}) \to {}_{L'}\mathbb{C}\mathsf{sp}(\mathsf{X})$ that acts*

- *on objects and arrows, as the identity;*
- *on proarrows $a \nrightarrow b$, by restricting the underlying cospan $L(a) \to x \leftarrow L(b)$ along the components $\alpha_a : L'(a) \to L(a)$ and $\alpha_b : L'(b) \to L(b)$;*
- *on cells $\begin{array}{ccc} a & \overset{m}{\nrightarrow} & b \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ c & \underset{n}{\nrightarrow} & d \end{array}$, by pasting the naturality squares for f and g:*

$$\begin{array}{ccccccccc} L'(a) & \xrightarrow{\alpha_a} & L(a) & \longrightarrow & x & \longleftarrow & L(b) & \xleftarrow{\alpha_b} & L'(b) \\ {\scriptstyle L'(f)}\downarrow & & {\scriptstyle L(f)}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle L(g)} & & \downarrow{\scriptstyle L'(g)} \\ L'(c) & \xrightarrow[\alpha_c]{} & L(c) & \longrightarrow & y & \longleftarrow & L(d) & \xleftarrow[\alpha_d]{} & L'(d) \end{array} .$$

The laxator $\alpha^*_{m,n} : \alpha^*(m) \odot \alpha^*(n) \to \alpha^*(m \odot n)$ for proarrows $m = (a,\ L(a) \to x \leftarrow L(b),\ b)$ and $n = (b,\ L(b) \to y \leftarrow L(c),\ c)$ has apex map given by the universal property of the pushout over $L'(b)$:

$$\begin{array}{ccccccc} & & & & y & & \\ & & & \nearrow & {\scriptstyle \iota'_y}\downarrow & \overset{\iota_y}{\searrow} & \\ L'(b) & \xrightarrow{\alpha_b} & L(b) & & x +_{L'(b)} y & \dashrightarrow & x +_{L(b)} y \\ & & & \searrow & {\scriptstyle \iota'_x}\uparrow & \underset{\iota_x}{\nearrow} & \\ & & & & x & & \end{array} .$$

The unitor $\alpha^*_a : \mathrm{id}'_a \to \alpha^*(\mathrm{id}_a)$ *for object* $a \in \mathsf{A}$ *has apex map* $\alpha_a : L'(a) \to L(a)$*. The lax double functor* α^* *is pseudo whenever* α *is a natural isomorphism, and it is automatically cocartesian whenever the structured cospan double categories are cocartesian.*

Proof. The laxators and unitors obey the coherence axioms by the uniqueness part of the universal property. Importantly, the last statement about cocartesianness holds because natural transformations automatically commute with coproducts. That is, if A and X have finite coproducts, then, using the notation of the proof of Theorem 2.3, the following diagrams commute for all objects $a,b \in \mathsf{A}$:

$$\begin{array}{ccc} L'(a)+L'(b) & \xrightarrow{L'_{a,b}} & L'(a+b) \\ {\scriptstyle \alpha_a+\alpha_b}\downarrow & & \downarrow{\scriptstyle \alpha_{a+b}} \\ L(a)+L(b) & \xrightarrow[L_{a,b}]{} & L(a+b) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & \overset{L'_0}{\nearrow} & L'(0_{\mathsf{A}}) \\ 0_{\mathsf{X}} & & \downarrow{\scriptstyle \alpha_0} \\ & \underset{L_0}{\searrow} & L(0_{\mathsf{A}}) \end{array} \ .$$

Restricting along the components of α thus commutes with restricting along the inverse comparison maps and so also commutes with coproducts of structured cospans. □

Proof of Theorem 2.4. Using the three lemmas, the lax double functor $\mathbb{F} : {}_L\mathbb{C}\mathsf{sp}(\mathsf{X}) \to {}_{L'}\mathbb{C}\mathsf{sp}(\mathsf{X}')$ is realized as the composite

$$\mathbb{F} : {}_L\mathbb{C}\mathsf{sp}(\mathsf{X}) \xrightarrow{{}_L\mathbb{C}\mathsf{sp}(F_1)} {}_{F_1 \circ L}\mathbb{C}\mathsf{sp}(\mathsf{X}') \xrightarrow{\alpha^*} {}_{L' \circ F_0}\mathbb{C}\mathsf{sp}(\mathsf{X}') \xrightarrow{{}_{F_0}\mathbb{C}\mathsf{sp}(\mathsf{X}')} {}_{L'}\mathbb{C}\mathsf{sp}(\mathsf{X}'). \qquad \square$$

3 Decorated cospans as a double Grothendieck construction

Decorated cospans represent open systems as cospans with apexes decorated by extra data [6, 12]. For example, open dynamical systems comprise a cospan of finite sets along with a dynamical system whose set of state variables is the apex set [7]. In contrast to structured cospans, the symmetric monoidal product of decorated cospans need not satisfy a universal property such as cocartesianness. Decorated cospans are therefore applicable in certain situations where structured cospans are not, at the expense of requiring more data to construct.

The Grothendieck construction $\int F$ of a functor $F : \mathsf{A} \to \mathbf{Cat}$ can be thought to decorate the objects of A with data from F, inasmuch as the objects of $\int F$ consist of an object $a \in \mathsf{A}$ together with an object $x \in F(a)$ (the "decoration"). So one might suppose that decorated cospans arise from a Grothendieck construction. For that to be the case, the cospans being decorated must be the *objects* of a category. Fortunately, as we emphasized in Section 1, that is precisely how cospans are seen by the modern theory of double categories. In this section, we reconstruct and generalize the double category of decorated cospans using the double-categorical analogue of the Grothendieck construction.

3.1 Double Grothendieck construction

In their study of double fibrations [10], Cruttwell, Lambert, Pronk, and Szyld introduced a Grothendieck construction for double categories, taking as input a lax double functor into $\mathbb{S}\mathsf{pan}(\mathbf{Cat})$.[3]

[3] In its most general form, the double Grothendieck construction takes as input a lax double *pseudo* functor into $\mathbb{S}\mathsf{pan}(\mathbf{Cat})$, analogous to how the Grothendieck construction takes a *pseudo*functor into **Cat**. For simplicity, we eschew this aspect but see [10, Definition 3.12].

Before stating the construction, we unpack some of the considerable amount of data contained in a lax double functor $F : \mathbb{A} \to \mathbb{S}\mathsf{pan}(\mathbf{Cat})$. First, there are natural transformations

$$\sigma : \mathrm{apex} \circ F_1 \Rightarrow F_0 \circ \mathrm{src} : \mathbb{A}_1 \to \mathbf{Cat} \qquad \text{and} \qquad \tau : \mathrm{apex} \circ F_1 \Rightarrow F_0 \circ \mathrm{tgt} : \mathbb{A}_1 \to \mathbf{Cat}$$

whose components are the functors σ_m and τ_m defined by

$$F_1(m) =: \Big(F_0(a) = \mathrm{ft}_L(F_1(m)) \xleftarrow{\sigma_m} \mathrm{apex}(F_1(m)) \xrightarrow{\tau_m} \mathrm{ft}_R(F_1(m)) = F_0(b) \Big)$$

for each proarrow $m : a \nrightarrow b$ in $\mathbb{A}$. The naturality squares for σ and τ are precisely the maps of spans

$$F_1(\alpha) = \left(\begin{array}{ccccccccc} F_0(a) & = & \mathrm{ft}_L(F_1(m)) & \xleftarrow{\sigma_m} & \mathrm{apex}(F_1(m)) & \xrightarrow{\tau_m} & \mathrm{ft}_R(F_1(m)) & = & F_0(b) \\ {\scriptstyle F_0(f)}\downarrow & & {\scriptstyle \mathrm{ft}_L(F_1(\alpha))}\downarrow & & {\scriptstyle \mathrm{apex}(F_1(\alpha))}\downarrow & & \downarrow{\scriptstyle \mathrm{ft}_R(F_1(\alpha))} & & \downarrow{\scriptstyle F_0(g)} \\ F_0(c) & = & \mathrm{ft}_L(F_1(n)) & \xleftarrow[\sigma_n]{} & \mathrm{apex}(F_1(n)) & \xrightarrow[\tau_n]{} & \mathrm{ft}_R(F_1(n)) & = & F_0(d) \end{array} \right)$$

for each cell $\begin{array}{ccc} a & \overset{m}{\nrightarrow} & b \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\ c & \underset{n}{\nrightarrow} & d \end{array}$ in $\mathbb{A}$. Writing $F_{m,n} : F(m) \odot F(n) \to F(m \odot n)$ and $F_a : \mathrm{id}_{Fa} \to F(\mathrm{id}_a)$ for the laxators and unitors of F, there are also natural families of functors

$$\Phi_{m,n} := \mathrm{apex}(F_{m,n}) : \mathrm{apex}(F(m)) \,{}_{\tau_m}\!\times_{\sigma_n} \mathrm{apex}(F(n)) \to \mathrm{apex}(F(m \odot n))$$

and $\Phi_a := \mathrm{apex}(F_a) : F(a) \to \mathrm{apex}(F(\mathrm{id}_x))$, indexed by proarrows $a \overset{m}{\nrightarrow} b \overset{n}{\nrightarrow} c$ and objects a in $\mathbb{A}$.

Using this notation, the double Grothendieck construction [10, Theorem 3.51] appears as:

Theorem 3.1. *Given a lax double functor $F : \mathbb{A} \to \mathbb{S}\mathsf{pan}(\mathbf{Cat})$, there is a double category $\int F$, the **double Grothendieck construction** of F, with underlying categories $(\int F)_0 = \int F_0$ and $(\int F)_1 = \int(\mathrm{apex} \circ F_1)$. Explicitly, the double category $\int F$ has*

- *as objects, pairs (a,x) where a is an object of $\mathbb{A}$ and x is an object of $F(a)$;*
- *as arrows $(a,x) \to (b,y)$, pairs (f,ϕ) where $f : a \to b$ is an arrow of $\mathbb{A}$ and $\phi : F(f)(x) \to y$ is a morphism of $F(b)$;*
- *as proarrows $(a,x) \nrightarrow (b,y)$, pairs (m,s) where $m : a \nrightarrow b$ is a proarrow of $\mathbb{A}$ and s is an object of* $\mathrm{apex}(F(m))$ *such that $\sigma_m(s) = x$ and $\tau_m(s) = y$;*
- *as cells* $\begin{array}{ccc} (a,x) & \overset{(m,s)}{\nrightarrow} & (b,y) \\ {\scriptstyle (f,\phi)}\downarrow & & \downarrow{\scriptstyle (g,\psi)} \\ (c,w) & \underset{(n,t)}{\nrightarrow} & (d,z) \end{array}$, *pairs (α,ν) such that* $\begin{array}{ccc} a & \overset{m}{\nrightarrow} & b \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\ c & \underset{n}{\nrightarrow} & d \end{array}$ *is a cell in $\mathbb{A}$ and $\nu : \mathrm{apex}(F(\alpha))(s) \to t$ is a morphism of* $\mathrm{apex}(F(n))$ *such that $\sigma_n(\nu) = \phi$ and $\tau_n(\nu) = \psi$.*

External composition and identities in $\int F$ are as follows.

- *The composite of proarrows $(a,x) \overset{(m,s)}{\nrightarrow} (b,y) \overset{(n,t)}{\nrightarrow} (c,z)$ is $(m \odot n, \Phi_{m,n}(s,t)) : (a,x) \nrightarrow (b,y)$.*
- *The external composite of cells is*

$$\begin{array}{ccccc} (a,x) & \xrightarrow{(m,s)} & (b,y) & \xrightarrow{(n,t)} & (c,z) \\ {\scriptstyle (f,\phi)}\downarrow & (\alpha,\mu) & {\scriptstyle (g,\psi)}\downarrow & (\beta,\nu) & \downarrow{\scriptstyle (h,\eta)} \\ (a',x') & \xrightarrow[(m',s')]{} & (b',y') & \xrightarrow[(n',t')]{} & (c',z') \end{array} \quad := \quad \begin{array}{ccc} (a,x) & \xrightarrow{(m\odot n, \Phi_{m,n}(s,t))} & (c,z) \\ {\scriptstyle (f,\phi)}\downarrow & (\alpha\odot\beta, \Phi_{m',n'}(\mu,\nu)) & \downarrow{\scriptstyle (h,\eta)} \\ (a',x') & \xrightarrow[(m'\odot n', \Phi_{m',n'}(s',t'))]{} & (c',z') \end{array} .$$

- *The identity proarrow at object* (a,x) *is* $(\mathrm{id}_a, \Phi_a(x))$.
- *The identity cell at arrow* $(f,\phi) : (a,x) \to (b,y)$ *is* $(\mathrm{id}_f, \Phi_b(\phi))$.

Moreover, there is a canonical ***projection*** $\pi_F : \int F \to \mathbb{A}$*, which is a strict double functor.*

3.2 A modular reconstruction of decorated cospans

To define decorated cospans, we apply the double Grothendieck construction in the case that the base double category $\mathbb{A}$ is a double category of cospans. Specifically, let A be a category with pushouts and let $F : \mathbb{C}\mathsf{sp}(\mathsf{A}) \to \mathbb{S}\mathsf{pan}(\mathbf{Cat})$ be a lax double functor. Then the **double category of F-decorated cospans**, denoted $F\mathbb{C}\mathsf{sp}$, is the double Grothendieck construction $\int F$.

This notion of decorated cospan is more general than the established one [6, §2] in two different ways. First, the decorations assigned to a cospan may depend on the whole cospan, not just on its apex. Second, the feet of the cospans receive their own decorations, which can be extracted from the cospan decorations using the transformations denoted σ and τ above. For two decorated cospans to be composable, not only must the feet of the cospans be compatible, so must be the decorations on the feet. We will see an application that takes advantage of this extra generality shortly. Before that, we show how to recover the original notion of decorated cospan based on lax monoidal functors into $(\mathbf{Cat}, \times)$.

Corollary 3.2. *Let* A *be a category with finite colimits and let* $F : (\mathsf{A}, +) \to (\mathbf{Cat}, \times)$ *be a lax monoidal functor. Then there is a double category* $F\mathbb{C}\mathsf{sp}$ *that has*

- *as objects, the objects of* A*;*
- *as arrows, the morphisms of* A*;*
- *as proarrows* $a \nrightarrow b$*,* F***-decorated cospans*** *with feet* a *and* b*, which are cospans* $p = (a \to m \leftarrow b)$ *in* A *together with a* ***decoration*** $s \in F(m)$*;*
- *as cells* $\begin{array}{ccc} a & \overset{(p,s)}{\nrightarrow} & b \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ c & \underset{(q,t)}{\nrightarrow} & d \end{array}$ *where* $p = (a \to m \leftarrow b)$ *and* $q = (c \to n \leftarrow d)$*,* ***morphisms of*** F***-decorated cospans*** *with foot maps* f *and* g*, which are morphisms of cospans in* A *of the form*

$$\begin{array}{ccccc} a & \longrightarrow & m & \longleftarrow & c \\ {\scriptstyle f}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ b & \longrightarrow & n & \longleftarrow & d \end{array}$$

together with a ***decoration morphism*** $\nu : F(h)(s) \to t$ *in* $F(n)$*.*

The composite of proarrows $a \overset{(p,s)}{\nrightarrow} b \overset{(q,t)}{\nrightarrow} c$*, where* $p = (a \to m \leftarrow b)$ *and* $q = (b \to n \leftarrow c)$*, is the proarrow* $(p \odot q, \Phi_{m,n}(s,t))$*, where the cospan* $p \odot q$ *is given by pushout in* A *and the functor* $\Phi_{m,n}$ *is the composite*

$$\Phi_{m,n} : F(m) \times F(n) \xrightarrow{F_{m,n}} F(m+n) \xrightarrow{F([\iota_m, \iota_n])} F(m +_b n).$$

The identity proarrow at $a \in \mathsf{A}$ *is* (id_a, Φ_a)*, where* Φ_a *is the composite* $1 \xrightarrow{F_0} F(0) \xrightarrow{F(!)} F(a)$*.*

Moreover, there is a canonical ***projection*** $\pi_F : F\mathbb{C}\mathsf{sp} \to \mathbb{C}\mathsf{sp}(\mathsf{A})$*, which is a strict double functor.*

Proof. We construct the double category $F\mathbb{C}\mathsf{sp}$ in a modular fashion by applying the double Grothendieck construction to a lax double functor $\tilde{F} : \mathbb{C}\mathsf{sp}(\mathsf{A}) \to \mathbb{S}\mathsf{pan}(\mathbf{Cat})$ that is itself the composite of three simpler lax double functors:

$$\tilde{F} : \mathbb{C}\mathsf{sp}(\mathsf{A}) \xrightarrow{\mathrm{Apex}} \mathbb{B}(\mathsf{A}, +) \xrightarrow{\mathbb{B}F} \mathbb{B}(\mathbf{Cat}, \times) \xrightarrow{\mathrm{Apex}_*} \mathbb{S}\mathsf{pan}(\mathbf{Cat}).$$

Let us explain each of these. First, any monoidal category $(\mathsf{C}, \otimes, I)$ can be regarded as a double category $\mathbb{D}$ whose category of objects is trivial, $\mathbb{D}_0 = 1$; whose category of morphisms is $\mathbb{D}_1 = \mathsf{C}$; and whose external composition and identity are the monoidal product and unit [14, §3.3.4]. Lax monoidal functors then induce lax double functors between such degenerate double categories, and monoidal natural transformations induce natural transformations of those, so altogether there is a 2-functor $\mathbb{B} : \mathbf{MonCat}_{\text{lax}} \to \mathbf{Dbl}_{\text{lax}}$. In particular, the lax monoidal functor $F : (\mathsf{A}, +) \to (\mathbf{Cat}, \times)$ induces a lax double functor $\mathbb{B}F$.

Next, given a category A with finite colimits, the lax double functor $\text{Apex} : \mathbb{C}\mathsf{sp}(\mathsf{A}) \to \mathbb{B}(\mathsf{A}, +)$ has the unique map $\text{Apex}_0 : \mathsf{A} \xrightarrow{!} 1$ between categories of objects and the functor $\text{Apex}_1 := \mathsf{A}^{\{\bullet \to \bullet \leftarrow \bullet\}} \xrightarrow{\text{apex}} \mathsf{A}$ between categories of morphisms. The laxators

$$\text{Apex}_{p,q} : \text{Apex}(p) + \text{Apex}(q) = m + n \xrightarrow{[\iota_m, \iota_n]} m +_b n = \text{Apex}(p \odot n)$$

for proarrows $p = (a \to m \leftarrow b)$ and $q = (b \to n \leftarrow c)$, and the unitors $\text{Apex}_a : 0 \xrightarrow{!} a$ for objects $a \in \mathsf{A}$, are all given by the universal properties of the colimits involved.

Finally, given a category C with finite limits, the double functor $\text{Apex}_* : \mathbb{B}(\mathsf{C}, \times) \to \mathbb{S}\mathsf{pan}(\mathsf{C})$ has underlying functors $(\text{Apex}_*)_0 : 1 \to \mathsf{C}$ picking out the terminal object 1 of C and $(\text{Apex}_*)_1 : \mathsf{C} \to \mathsf{C}^{\{\bullet \leftarrow \bullet \to \bullet\}}$ sending each object $c \in \mathsf{C}$ to the span $1 \xleftarrow{!} c \xrightarrow{!} 1$. This double functor is pseudo because products are isomorphic to pullbacks over the terminal object. By making reasonable choices of products and pullbacks, we can even assume that the double functor is strict.

The double category $F\mathbb{C}\mathsf{sp}$ is precisely the double Grothendieck construction of $\tilde{F}$ (Theorem 3.1). This follows from the formulas for the laxators and unitors of a composite lax double functor [14, Equation 3.63]. In terms of the notation in the corollary statement, the laxators and unitors of the composite $\tilde{F}$ are $\tilde{F}_{p,q} = \text{Apex}_*(\Phi_{\text{Apex}(p),\text{Apex}(q)})$ and $\tilde{F}_a = \text{Apex}_*(\Phi_a)$. □

This result was first proved in [6, Theorem 2.1]. Our reconstruction solves a lingering conceptual puzzle about the composition law for decorated cospans: why does it involve two operations, instead of just one? As the proof shows, the reason is that decorated cospans implicitly use a *composite* of lax double functors. Specifically, laxators from the lax monoidal functor F combine with laxators from the lax double functor Apex to give the distinctive formula for composing decorations of decorated cospans.

3.3 Application: double category of process theories

An early and recurring theme of applied category theory is the mathematical modeling of physical or computational processes by monoidal categories, often with extra structure [4]. To describe a process syntactically, one can define, say by generators and relations, a small category T with the relevant structure, and then choose a particular morphism p in T. The category T defines the basic material for the process and the morphism p specifies the process itself. Regarding the category T as a theory in the sense of the categorical logic, the pair (T, p) might be called a *theory of a process*, or *process theory* for short. For example, in the author's thesis [20], a *statistical theory* is defined to be a small Markov category [13] equipped with extra linear algebraic structure, together with a distinguished morphism $p : \theta \to x$ representing the data generating process for a statistical model.

To be more precise, process theories are defined relative to a **concrete 2-category**, by which we mean a 2-category $\mathbf{C}$ equipped with a 2-functor $|-| : \mathbf{C} \to \mathbf{Cat}$, giving the **underlying category** of $\mathbf{C}$. This 2-functor will often satisfy additional properties, such as being locally faithful, but we need not assume that. Given a morphism $F : \mathsf{X} \to \mathsf{Y}$ in a concrete 2-category, we will write $F(x) := |F|(x)$ and $F(f) := |F|(f)$ for the action of the underlying functor of F on the objects and morphisms of $|\mathsf{X}|$. As an example, statistical theories are based on the concrete 2-category of small linear algebraic Markov categories, structure-preserving monoidal functors, and monoidal natural transformations [20].

Process theories can be composed once their underlying theories are made open. In the context of statistics, this composition corresponds to making hierarchical statistical models, where samples from one model become parameters of the next. To express this mathematically, we construct a double category of process theories. We need two main ingredients: the double Grothendieck construction, and an extension of the familiar construction of comma categories to a lax double functor. We now review the latter, which is interesting in its own right.

There is a lax double functor Comma : $\mathbb{C}\mathsf{sp}(\mathbf{Cat}) \to \mathbb{S}\mathsf{pan}(\mathbf{Cat})$ that is the identity on objects and arrows and sends a cospan of categories $(\mathsf{A} \xrightarrow{i} \mathsf{X} \xleftarrow{o} \mathsf{B})$ to the span of categories $(\mathsf{A} \xleftarrow{\pi_\mathsf{A}} i/o \xrightarrow{\pi_\mathsf{B}} \mathsf{B})$ comprising the comma category i/o with its canonical projections.[4] It acts on maps of cospans as

$$\begin{array}{ccccc} \mathsf{A} & \xrightarrow{i} & \mathsf{X} & \xleftarrow{o} & \mathsf{B} \\ {\scriptstyle H}\downarrow & & {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle K} \\ \mathsf{A}' & \xrightarrow[i']{} & \mathsf{X}' & \xleftarrow[o']{} & \mathsf{B}' \end{array} \quad\mapsto\quad \begin{array}{ccccc} \mathsf{A} & \xleftarrow{\pi_\mathsf{A}} & i/o & \xrightarrow{\pi_\mathsf{B}} & \mathsf{B} \\ {\scriptstyle H}\downarrow & & {\scriptstyle \tilde{F}}\downarrow & & \downarrow{\scriptstyle K} \\ \mathsf{A}' & \xleftarrow[\pi_{\mathsf{A}'}]{} & i'/o' & \xrightarrow[\pi_{\mathsf{B}'}]{} & \mathsf{B}' \end{array},$$

where the functor denoted $\tilde{F}$ sends an object $(a,\, i(a) \xrightarrow{f} o(b),\, b)$ of the comma category i/o to

$$(H(a),\, i'(H(a)) = F(i(a)) \xrightarrow{F(f)} F(o(b)) = o'(H(b)),\, H(b))$$

and a morphism (h,k) to $(H(h),K(k))$.

To describe the laxators, let $m = (\mathsf{A} \xrightarrow{i} \mathsf{X} \xleftarrow{o} \mathsf{B})$ and $n = (\mathsf{B} \xrightarrow{j} \mathsf{Y} \xleftarrow{p} \mathsf{C})$ be composable cospans of categories and let $\iota_\mathsf{X} : \mathsf{X} \to \mathsf{X} +_\mathsf{B} \mathsf{Y}$ and $\iota_\mathsf{Y} : \mathsf{Y} \to \mathsf{X} +_\mathsf{B} \mathsf{Y}$ be the inclusions into the pushout of categories. Then the apex map of the laxator $\mathrm{Comma}_{m,n}$ is the functor

$$(i/o) \times_\mathsf{B} (j/p) \to (\iota_\mathsf{X} \circ i)/(\iota_\mathsf{Y} \circ p)$$

that sends a pair of objects (a,f,b) and (b,g,c) with $o(b) = j(b)$ to $(a, \iota_\mathsf{Y}(g) \circ \iota_\mathsf{X}(f), c)$, which is well-defined since $\iota_\mathsf{X}(o(b)) = \iota_\mathsf{Y}(j(b))$ in $\mathsf{X} +_\mathsf{B} \mathsf{Y}$. This functor sends a pair of maps (h,k) and (k,ℓ) to the map (h,ℓ). Finally, given a category A, the apex map of the unitor $\mathrm{Comma}_\mathsf{A}$ is the functor $\mathsf{A} \to 1_\mathsf{A}/1_\mathsf{A}$ that sends an object $a \in \mathsf{A}$ to $(a, 1_a, a)$ and a morphism h to (h,h).

Proposition 3.3. *Let* $\mathbf{C}$ *be a concrete 2-category with pushouts. Then there is a double category that has*

- *as objects, an object* A *in* $\mathbf{C}$ *together with an object* $a \in |\mathsf{A}|$*;*
- *as arrows* $(\mathsf{A},a) \to (\mathsf{A}',a')$*, a morphism* $H : \mathsf{A} \to \mathsf{A}'$ *in* $\mathbf{C}$ *together with a morphism* $h' : H(a) \to a'$ *in* $|\mathsf{A}'|$*;*
- *as proarrows* $(\mathsf{A},a) \nrightarrow (\mathsf{B},b)$*, a cospan in* $\mathbf{C}$ *of form* $m = (\mathsf{A} \xrightarrow{i} \mathsf{X} \xleftarrow{o} \mathsf{B})$ *along with a morphism* $f : i(a) \to o(b)$ *in* $|\mathsf{X}|$*;*

[4] The lax double functor Comma : $\mathbb{C}\mathsf{sp}(\mathbf{C}) \to \mathbb{S}\mathsf{pan}(\mathbf{C})$ even generalizes from $\mathbf{C} = \mathbf{Cat}$ to any 2-category $\mathbf{C}$ with comma objects, pushouts, and pullbacks [14, §4.5.9], although we will not use that.

- *as cells* $\begin{array}{ccc} (\mathsf{A},a) & \overset{(m,f)}{\rightarrow\!\!\!\!\!+} & (\mathsf{B},b) \\ {\scriptstyle (H,h')}\downarrow & & \downarrow{\scriptstyle (K,k')} \\ (\mathsf{A}',a') & \underset{(m',f')}{\rightarrow\!\!\!\!\!+} & (\mathsf{B}',b') \end{array}$, *a morphism* $F : \mathsf{X} \to \mathsf{X}'$ *forming a map of cospans* $\begin{array}{ccccc} \mathsf{A} & \xrightarrow{i} & \mathsf{X} & \xleftarrow{o} & \mathsf{B} \\ {\scriptstyle H}\downarrow & & {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle K} \\ \mathsf{A}' & \underset{i'}{\rightarrow} & \mathsf{X}' & \underset{o'}{\leftarrow} & \mathsf{B}' \end{array}$

  in $\mathbf{C}$ *and making the following square in* $|\mathsf{X}'|$ *commute:*

$$\begin{array}{ccc} i'(H(a)) = F(i(a)) \overset{F(f)}{\longrightarrow} F(o(b)) = o'(K(b)) \\ {\scriptstyle i'(h')}\downarrow \qquad\qquad\qquad\qquad \downarrow{\scriptstyle o'(k')} \\ i'(a') \xrightarrow{\qquad\qquad f' \qquad\qquad} o'(b') \end{array} .$$

Two proarrows $(\mathsf{A},a) \overset{(m,f)}{\nrightarrow} (\mathsf{B},b) \overset{(n,g)}{\nrightarrow} (\mathsf{C},c)$, *with* $m = (\mathsf{A} \xrightarrow{i} \mathsf{X} \xleftarrow{o} \mathsf{B})$ *and* $n = (\mathsf{B} \xrightarrow{j} \mathsf{Y} \xleftarrow{p} \mathsf{C})$, *have composite* $(m \odot n, h) : (\mathsf{A},a) \nrightarrow (\mathsf{C},c)$, *where* $m \odot n$ *is the composite cospan in* $\mathbf{C}$ *with apex* $\mathsf{X} +_{\mathsf{B}} \mathsf{Y}$ *and* h *is given by first composing the images of* f *and* g *in* $|\mathsf{X}| +_{|\mathsf{B}|} |\mathsf{Y}|$ *and then applying the canonical functor* $|\mathsf{X}| +_{|\mathsf{B}|} |\mathsf{Y}| \to |\mathsf{X} +_{\mathsf{B}} \mathsf{Y}|$. *The identity proarrow at* (A,a) *is* $(\mathrm{id}_{\mathsf{A}}, 1_a)$.

Proof. Apply the double Grothendieck construction to the composite lax double functor

$$\mathbb{C}\mathsf{sp}(\mathbf{C}) \xrightarrow{\mathbb{C}\mathsf{sp}(|-|)} \mathbb{C}\mathsf{sp}(\mathbf{Cat}) \xrightarrow{\text{Comma}} \mathbb{S}\mathsf{pan}(\mathbf{Cat}).$$

Here the lax double functor $\mathbb{C}\mathsf{sp}(|-|)$ is a particular case of Example A.5. □

4 Conclusion

We have revisited structured and decorated cospans from the perspective of double category theory, showing that double categories of structured cospans form cocartesian equipments and that their maps are cocartesian double functors. We have also reconstructed and generalized double categories of decorated cospans using the double Grothendieck construction.

Looking to future developments, we have presented a reasonably complete and self-contained treatment of the theory of structured cospans, but less so for the theory of decorated cospans. We have not shown how to construct maps between double categories of decorated cospans, along the lines of Baez et al's [6, Theorem 2.5]. Just as the classical Grothendieck construction for categories is 2-functorial [21, §6], so should be the Grothendieck construction for double categories, which should in turn directly produce maps between decorated cospan double categories and natural transformations between those. Equally importantly, we have not recovered the symmetric monoidal product of decorated cospans, an absence clearly felt in our example of the double category of process theories. Monoidal products should be obtained as a corollary of a hypothetical Grothendieck construction for monoidal double categories, combining the monoidal and double Grothendieck constructions [10, 19]. In these and other ways, we expect the further development of the theory of double categories to immediately impact the study of open systems, simplifying known constructions, suggesting new ones, and enabling practitioners to focus on applications rather than general theoretical issues.

Acknowledgments This project was partially supported by the Air Force Office of Scientific Research (AFOSR) Young Investigator Program (YIP) through Award FA9550-23-1-0133. I thank Nathanael Arkor, John Baez, and Brandon Shapiro for helpful conversations. I am also grateful to Brandon Shapiro for comments on early versions of this work.

References

[1] Rebekah Aduddell, James Fairbanks, Amit Kumar, Pablo S. Ocal, Evan Patterson & Brandon T. Shapiro (2023): *A compositional account of motifs, mechanisms, and dynamics in biochemical regulatory networks*, arXiv: 2301.01445.

[2] Evangelia Aleiferi (2018): *Cartesian double categories with an emphasis on characterizing spans*, PhD thesis. Dalhousie University, arXiv: 1809.06940.

[3] John Baez, Xiaoyan Li, Sophie Libkind, Nathaniel Osgood & Evan Patterson (2022): *Compositional modeling with stock and flow diagrams*. In *Proceedings of the Applied Category Theory Conference 2022*, arXiv: 2205.08373. In press.

[4] John Baez & Mike Stay (2010): *Physics, topology, logic and computation: a Rosetta Stone*. In *New structures for physics*, pages 95–172. doi: 10.1007/978-3-642-12821-9_2, arXiv: 0903.0340.

[5] John C. Baez & Kenny Courser (2020): *Structured cospans*. *Theory and Applications of Category Theory*. 35(48), pages 1771–1822, arXiv: 1911.04630. Available at http://www.tac.mta.ca/tac/volumes/35/48/35-48abs.html.

[6] John C. Baez, Kenny Courser & Christina Vasilakopoulou (2022): *Structured versus decorated cospans*. *Compositionality*. 4(3), doi: 10.32408/compositionality-4-3, arXiv: 2101.09363.

[7] John C. Baez & Blake S. Pollard (2017): *A compositional framework for reaction networks*. *Reviews in Mathematical Physics*. 29(09), page 1750028, doi: 10.1142/S0129055X17500283, arXiv: 1704.02051.

[8] Aurelio Carboni, G. Max Kelly & Richard J. Wood (1991): *A 2-categorical approach to change of base and geometric morphisms I*. *Cahiers de topologie et géométrie différentielle catégoriques*. 32(1), pages 47–95.

[9] Kenny Courser (2020): *Open systems: A double categorical perspective*, PhD thesis. University of California, Riverside, arXiv: 2008.02394.

[10] Geoffrey Cruttwell, Michael Lambert, Dorette Pronk & Martin Szyld (2022): *Double Fibrations*. *Theory and Applications of Categories*. 38(35), pages 1326–1394, arXiv: 2205.15240. Available at http://www.tac.mta.ca/tac/volumes/38/35/38-35abs.html.

[11] José Luiz Fiadeiro & Vincent Schmitt (2007): *Structured co-spans: an algebra of interaction protocols*. In *International Conference on Algebra and Coalgebra in Computer Science (CALCO 2007)*, pages 194–208, doi: 10.1007/978-3-540-73859-6_14.

[12] Brendan Fong (2015): *Decorated cospans*. *Theory and Applications of Categories*. 30(33), pages 1096–1120, arXiv: 1502.00872. Available at http://www.tac.mta.ca/tac/volumes/30/33/30-33abs.html.

[13] Tobias Fritz (2020): *A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics*. *Advances in Mathematics*. 370, page 107239, doi: 10.1016/j.aim.2020.107239, arXiv: 1908.07021.

[14] Marco Grandis (2019): *Higher dimensional categories: From double to multiple categories*. World Scientific. doi: 10.1142/11406.

[15] Marco Grandis & Robert Paré (1999): *Limits in double categories*. *Cahiers de topologie et géométrie différentielle catégoriques*. 40(3), pages 162–220. Available at http://www.numdam.org/item/CTGDC_1999__40_3_162_0/.

[16] Linde Wester Hansen & Michael Shulman (2019): *Constructing symmetric monoidal bicategories functorially*, arXiv: 1910.09240.

[17] Piergiulio Katis, Nicoletta Sabadini & Robert F. C. Walters (1997): *Span(Graph): A categorical algebra of transition systems*. In *Algebraic Methodology and Software Technology. AMAST 1997*, Springer, pages 307–321, doi: 10.1007/BFb0000479.

[18] Sophie Libkind, Andrew Baas, Micah Halter, Evan Patterson & James P. Fairbanks (2022): *An algebraic framework for structured epidemic modelling*. *Philosophical Transactions of the Royal Society A*. 380(2233), page 20210309, doi: 10.1098/rsta.2021.0309, arXiv: `2203.16345`.

[19] Joe Moeller & Christina Vasilakopoulou (2020): *Monoidal Grothendieck construction*. *Theory and Applications of Categories*. 35(31), pages 1159–1207, arXiv: `1809.00727`. Available at `http://www.tac.mta.ca/tac/volumes/35/31/35-31abs.html`.

[20] Evan Patterson (2020): *The algebra and machine representation of statistical models*, PhD thesis. Stanford University, arXiv: `2006.08945`.

[21] George Peschke & Walter Tholen (2020): *Diagrams, fibrations, and the decomposition of colimits*, arXiv: `2006.10890`.

[22] Robert Rosebrugh, Nicoletta Sabadini & Robert F. C. Walters (2005): *Generic commutative separable algebras and cospans of graphs*. *Theory and Applications of Categories*. 15(6), pages 164–177. Available at `http://www.tac.mta.ca/tac/volumes/15/6/15-06abs.html`.

[23] Michael Shulman (2008): *Framed bicategories and monoidal fibrations*. *Theory and Applications of Categories*. 20(18), pages 650–738, arXiv: `0706.1286`. Available at `http://www.tac.mta.ca/tac/volumes/20/18/20-18abs.html`.

[24] Michael Shulman (2010): *Constructing symmetric monoidal bicategories*, arXiv: `1004.0993`.

A Cocartesian equipments

In this appendix, we review cocartesian double categories and equipments, and the maps between them. This material is known but may not be straightforward to access in the literature. It is included for the reader's convenience.

Just as a cocartesian category is (on one standard definition) a category with finite coproducts, a cocartesian double category is a double category with finite double-categorical coproducts. A highly conceptual way to make this precise is to define a cocartesian double category to be a cocartesian object in the 2-category **Dbl** of double categories, double functors, and natural transformations. Thus, a double category $\mathbb{D}$ is **cocartesian** if the diagonal and terminal double functors, $\Delta_{\mathbb{D}} : \mathbb{D} \to \mathbb{D} \times \mathbb{D}$ and $!_{\mathbb{D}} : \mathbb{D} \to \mathbb{1}$, have left adjoints in **Dbl**. This is (dual to) the approach taken by Aleiferi in her PhD thesis on cartesian double categories [2]. It will be convenient for us to have a more concrete description.[5]

Definition A.1. A double category $\mathbb{D}$ is **cocartesian** if its underlying categories $\mathbb{D}_0$ and $\mathbb{D}_1$ have finite coproducts; the source and target functors $\mathrm{src}, \mathrm{tgt} : \mathbb{D}_1 \rightrightarrows \mathbb{D}_0$ preserve finite coproducts; and the external composition $\odot : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$ and unit $\mathrm{id} : \mathbb{D}_0 \to \mathbb{D}_1$ also preserve finite coproducts, meaning that for all proarrows $x \overset{m}{\nrightarrow} y \overset{n}{\nrightarrow} z$ and $x' \overset{m'}{\nrightarrow} y' \overset{n'}{\nrightarrow} z'$ and objects x and x' in $\mathbb{D}$, the canonical comparison cells

$$\begin{array}{ccc} x+x' & \xrightarrow[\quad]{(m\odot n)+(m'\odot n')} & z+z' \\ \| & [\iota_m\odot\iota_n, \iota_{m'}\odot\iota_{n'}] & \| \\ x+x' & \xrightarrow[(m+m')\odot(n+n')]{} & z+z' \end{array} \quad \text{and} \quad \begin{array}{ccc} x+x' & \xrightarrow{\mathrm{id}_x+\mathrm{id}_{x'}} & x+x' \\ \| & [\mathrm{id}_{\iota_x}, \mathrm{id}_{\iota_{x'}}] & \| \\ x+x' & \xrightarrow[\mathrm{id}_{x+x'}]{} & x+x' \end{array}$$

[5] The equivalence of the two definitions follows from a general result about double adjunctions [14, Corollary 4.3.7].

given by the universal property of binary coproducts, as well as the comparison cells $0_{\mathbb{D}_1} \xrightarrow{!} 0_{\mathbb{D}_1} \odot 0_{\mathbb{D}_1}$ and $0_{\mathbb{D}_1} \xrightarrow{!} \mathrm{id}_{0_{\mathbb{D}_0}}$ given by the universal property of initial objects, are all isomorphisms in $\mathbb{D}_1$.

An equipment, also known as a fibrant double category or a framed bicategory, is a double category in which proarrows can be restricted or extended along pairs of arrows in a universal way. Equipments can be defined in at least three equivalent ways [23, Theorem 4.1], including as follows.

Definition A.2. An **equipment** is a double category $\mathbb{D}$ such that the pairing of the source and target functors, $\langle s,t\rangle : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$, is a fibration.

Elaborating the definition, a double category $\mathbb{D}$ is an equipment if every niche in $\mathbb{D}$ of the form on the left can be completed to a cell as on the right

$$
\begin{array}{ccc}
x & & y \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
w & \underset{h}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & z
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccc}
x & \xrightarrow{\mathrm{res}_g^f(n)} & y \\
{\scriptstyle f}\downarrow & \mathrm{res} & \downarrow{\scriptstyle g} \\
w & \underset{h}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & z
\end{array}
$$

called a **restriction** cell, with the universal property that for every pair of arrows $h : x' \to x$ and $k : y' \to y$, each cell α of the form on the left factors uniquely through the restriction cell as on the right:

$$
\begin{array}{ccc}
x' & \xrightarrow{m'} & y' \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\
x & \alpha & y \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
w & \underset{h}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & z
\end{array}
\quad = \quad
\begin{array}{ccc}
x' & \xrightarrow{m'} & y' \\
{\scriptstyle h}\downarrow & \exists! & \downarrow{\scriptstyle k} \\
x & \xrightarrow{\mathrm{res}_g^f(n)} & y \\
{\scriptstyle f}\downarrow & \mathrm{res} & \downarrow{\scriptstyle g} \\
w & \underset{h}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & z
\end{array}
$$

Finally, a **cocartesian equipment** is a double category that is both cocartesian and an equipment. We emphasize again that being a cocartesian equipment is a property of, not a structure on, a double category.

Example A.3 (Cospan double categories)**.** The prototypical example of a cocartesian equipment is none other than $\mathbb{C}\mathsf{sp}(\mathsf{S})$, the double category of cospans in a category S with finite colimits. Let us sketch the proof behind this statement. For a more detailed proof, one can dualize the proof in Aleiferi's thesis that $\mathbb{S}\mathsf{pan}(\mathsf{S})$, for a category S with finite limits, is a cartesian equipment [2].

Finite coproducts in the category $\mathbb{C}\mathsf{sp}(\mathsf{S})_0 = \mathsf{S}$ exist by assumption, and finite coproducts in the functor category $\mathbb{C}\mathsf{sp}(\mathsf{S})_1 = \mathsf{S}^{\{\bullet\to\bullet\leftarrow\bullet\}}$ are computed pointwise in S. So the source and target functors $\mathrm{ft}_L, \mathrm{ft}_R : \mathbb{C}\mathsf{sp}(\mathsf{S})_1 \to \mathsf{S}$, extracting the left and right feet, preserve coproducts. The comparison cells are isomorphisms because colimits commute with colimits (specifically, pushouts commute with coproducts) up to canonical isomorphism. Thus, the double category of cospans is cocartesian.

It is also an equipment. To restrict a cospan $c \xrightarrow{\ell} y \xleftarrow{r} d$ along a pair of morphisms $f : a \to c$ and $g : b \to d$, simply compose the morphisms with the legs of the cospan. The restriction cell is trivial:

$$
\begin{array}{ccccc}
a & \xrightarrow{\ell\circ f} & y & \xleftarrow{r\circ g} & b \\
{\scriptstyle f}\downarrow & & \| & & \downarrow{\scriptstyle g} \\
c & \underset{\ell}{\longrightarrow} & y & \underset{r}{\longleftarrow} & d
\end{array}
.
$$

We turn now to maps between cocartesian double categories and equipments. Since a cocartesian category is a cocartesian object in **Dbl**, a map between cocartesian double categories can be defined abstractly as a cocartesian morphism between cocartesian objects [8, §5.2]. As before, this definition reduces to a more concrete one:

Definition A.4. A double functor $F : \mathbb{D} \to \mathbb{E}$ between cocartesian double categories is **cocartesian** if both underlying functors $F_0 : \mathbb{D}_0 \to \mathbb{E}_0$ and $F_1 : \mathbb{D}_1 \to \mathbb{E}_1$ preserve finite coproducts.

Note that we will apply this definition to lax as well as pseudo double functors.

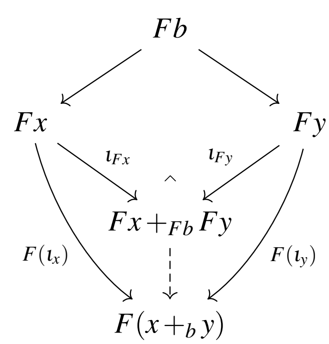
Perhaps surprisingly, no extra conditions on double functors between equipments are required. Any (op)lax double functor between equipments automatically preserves restriction (respectively, extension) cells, as proved by Shulman [23, Proposition 6.4]. In particular, a pseudo double functor between equipments preserves all the operations afforded by an equipment.

Example A.5 (Maps between cospan double categories)**.** The construction of the double category of cospans $\mathbb{C}\mathsf{sp}(\mathsf{S})$ extends to a 2-functor $\mathbb{C}\mathsf{sp} : \mathbf{Cat}_{\mathrm{po}} \to \mathbf{Dbl}_{\mathrm{lax}}$, where $\mathbf{Cat}_{\mathrm{po}}$ is the 2-category of categories with chosen pushouts, arbitrary functors, and natural transformations and $\mathbf{Dbl}_{\mathrm{lax}}$ is the 2-category of double categories, lax double functors, and natural transformations [14, §C3.11].

Let us describe the lax double functor $\mathbb{C}\mathsf{sp}(F) : \mathbb{C}\mathsf{sp}(\mathsf{S}) \to \mathbb{C}\mathsf{sp}(\mathsf{S}')$ induced by a functor $F : \mathsf{S} \to \mathsf{S}'$ between categories with pushouts. We have $\mathbb{C}\mathsf{sp}(F)_0 = F$ on objects and arrows, while $\mathbb{C}\mathsf{sp}(F)_1$ post-composes with F the diagrams defining cospans and maps of cospans in S. Since functors preserve identities, $\mathbb{C}\mathsf{sp}(F)$ is a *normal* lax double functor, meaning that it preserves identity proarrows strictly. Given cospans $m = (a \to x \leftarrow b)$ and $n = (b \to y \leftarrow c)$ in S, the laxator

$$\mathbb{C}\mathsf{sp}(F)_{m,n} : \mathbb{C}\mathsf{sp}(F)(m) \odot \mathbb{C}\mathsf{sp}(F)(n) \to \mathbb{C}\mathsf{sp}(F)(m \odot n)$$

has apex map given by the universal property of the pushout in S':



.

Clearly, $\mathbb{C}\mathsf{sp}(F)$ is pseudo if and only if F preserves pushouts.

Suppose that S and S' have all finite colimits, so that their double categories of cospans are cocartesian. Since coproducts of cospans are computed pointwise, $\mathbb{C}\mathsf{sp}(F)$ is a cocartesian lax double functor exactly when F preserves finite coproducts. Altogether, $\mathbb{C}\mathsf{sp}(F)$ is a cocartesian (pseudo) double functor if and only if F preserves all finite colimits.