Simplicial distributions, convex categories, and contextuality

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A striking phenomenon in physics, known as Bell's nonlocality and its generalization called contextuality, can be expressed as the nonexistence of a joint probability distribution over the set of all measurements that marginalizes to the distributions of the restricted set of measurements obtained from the experiment. Such a joint distribution always exists in classical theories. In particular, a joint distribution provides a model where all measurement outcomes are assigned before the measurement takes place, and the measurement probabilities are obtained by considering all such global assignments with a certain probability. It is a celebrated result of Bell [1] that in quantum theory, the joint distribution does not always exist, i.e., there are contextual families of distributions.

There are various approaches to formalizing contextuality. The presheaf approach is introduced in [2]. The ingredients in this approach are (1) a simplicial complex Σ whose vertices represent the set of all measurements, and its simplices are those that can be jointly performed, and (2) a finite set of outcomes for the measurements. For the outcome set, we don't lose any generality by considering the ring $\mathbb{Z}_d$ of integers modulo d. A distribution on $(\Sigma, \mathbb{Z}_d)$ is a presheaf of distributions, i.e., a family $(p_\sigma)_{\sigma\in\Sigma}$ together with a compatibility condition. Each p_σ is a distribution on the set of all functions $\sigma \to \mathbb{Z}_d$. More formally, let D_R denote the distribution monad on the category of sets [3], where R is a commutative semiring. Then p_σ belongs to $D_R(\mathbb{Z}_d^\sigma)$ where $R = \mathbb{R}_{\geq 0}$, the semiring of nonnegative reals. Another approach to contextuality is the topological approach of [4] based on techniques from the cohomology of groups. This approach introduces cohomology classes that can detect strong contextuality but fail to capture weaker versions, e.g., the famous Clauser–Horne–Shimony–Holt (CHSH) scenario [5]. In [6] both approaches are unified under the theory of *simplicial distributions*. This theory goes beyond the usual assumption that measurements and outcomes are represented by finite sets. In this framework, one can study distributions on spaces of measurements and outcomes, where a space is represented by a simplicial set. Simplicial sets are combinatorial objects more expressive than simplicial complexes. They are fundamental objects in modern homotopy theory [7]. A simplicial distribution is defined on a pair (X, Y) of simplicial sets, where X represents the measurements and Y the outcomes. The distribution monad can be elevated to a monad on the category of simplicial sets. Given Y, one can define another simplicial set $D_R(Y)$ whose simplices are distributions on the set of simplices of Y. A simplicial distribution is a morphism of simplicial sets $p : X \to D_R(Y)$.

In this paper, we study simplicial distributions from a categorical perspective. For a semiring R the algebras of the distribution monad $D_R : \mathbf{Set} \to \mathbf{Set}$ are called R-convex sets. This notion generalizes the usual notion of convexity for $R = \mathbb{R}_{\geq 0}$. Let $\mathbf{Conv}_R$ denote the category of R-convex sets. Every monad has an associated Kleisli category. In the case of D_R, the morphisms of the Kleisli category $s\mathbf{Set}_{D_R}$ are the simplicial distributions, i.e., for simplicial sets X, Y the set $\mathbf{Set}_{D_R}(X, Y)$ of morphisms is given by simplicial set morphisms $X \to D_R(Y)$. Let $s\mathbf{Conv}_R$ denote the category of simplicial R-convex sets. The main examples of such simplicial objects are $D_R(Y)$ for some simplicial set Y.

Proposition 0.1. *The functor sending a pair (X, Y) of simplicial sets to the set $s\mathbf{Set}(X, Y)$ of simplicial set morphisms restricts to a functor $s\mathbf{Set}(-,-) : s\mathbf{Set}^{op} \times s\mathbf{Conv}_R \to \mathbf{Conv}_R$.*

The main application of this result is to the set of simplicial distributions. By this result the set $s\mathbf{Set}(X, D_R(Y))$ is an R-convex set. Contextuality for simplicial distributions is defined using a comparison map $\Theta_{X,Y} : D_R(s\mathbf{Set}(X, Y)) \to s\mathbf{Set}(X, D_R(Y))$. Under this map, the delta distribution at a simplicial set morphism $\varphi : X \to Y$ is sent to the simplicial distribution given by the composition $X \xrightarrow{\varphi} Y \xrightarrow{\delta_Y} D_R(Y)$, called a deterministic distribution. Since the target is R-convex, $\Theta_{X,Y}$ is the unique extension to the free D_R-algebra, the domain of the map. A simplicial distribution is called contextual if it lies in the image of $\Theta_{X,Y}$; otherwise called noncontextual. This notion generalizes the notion of contextuality originally introduced in [2] for presheaves of distributions.

A category-theoretic point of view suggests lifting our analysis from the level of morphism sets to the level of categories. For this, we introduce the notion of *convex categories*. The first step is to lift D_R to a monad on the category $\mathbf{Cat}$ of (locally small) categories. Then we define an R-convex category as a D_R-algebra in $\mathbf{Cat}$. Every category enriched over the category of R-convex sets is an R-convex category. However, the converse does not always hold. The prominent example of a convex category is the Kleisli category $\mathbf{Set}_{D_R}$, and its simplicial version $s\mathbf{Set}_{D_R}$, which are not enriched over $\mathbf{Conv}_R$. We can think of $\Theta_{X,Y}$ assembled into a morphism in $\mathbf{ConvCat}_R$ from the free R-convex category to the Kleisli category, both of which obtained from the category of simplicial sets:

$$\Theta : D_R(s\mathbf{Set}) \to s\mathbf{Set}_{D_R}.$$

For outcome spaces which also have a group structure the convex set of simplicial distributions can be given a monoid structure. Our categorical framework combined with this monoid structure has interesting applications connecting contextuality to a weak notion of invertibility in convex monoids. Any presheaf of distributions can be realized as a simplicial distribution $p : X \to D_R(N\mathbb{Z}_d)$, where $N\mathbb{Z}_d$ is the nerve of the additive group $\mathbb{Z}_d$. The simplicial set $N\mathbb{Z}_d$ is, in fact, a simplicial group. The group structure on the nerve induces a monoid structure on the convex set $s\mathbf{Set}(X, D_R(N\mathbb{Z}_d))$ of simplicial distributions. We remark that this monoid structure is new and has not been investigated in the study of contextuality. More precisely, $s\mathbf{Set}(X, D_R(N\mathbb{Z}_d))$ is a convex monoid, i.e., a convex category with a single object. For a convex monoid M, with the map $\pi^M : D_R(M) \to M$ giving the D_R-algebra structure, an element $m \in M$ is called weakly invertible if it lies in the image of the composite $D_R(M^*) \xhookrightarrow{D_R(i_M)} D_R(M) \xrightarrow{\pi^M} M$, where $i_M : M^* \hookrightarrow M$ is the inclusion of the units. Our main result connects weak invertibility to noncontextuality.

Theorem 0.2. *Let R be a zero-sum-free, integral (commutative) semiring. Given a simplicial set X and a simplicial group Y, a distribution $p \in s\mathbf{Set}(X, D_R(Y))$ is noncontextual if and only if p is weakly invertible.*

Using our categorical approach, we also make the following contributions: (1) We introduce the notion of strong invertibility, a monoid-theoretic analogue of strong contextuality. (2) We introduce a degree of invertibility called invertible fraction which generalizes the notion of noncontextual fraction introduced in [2, 8]. (3) We show that simplicial homotopy can be used to detect extremal contextual distributions, a question of fundamental importance in the study of polytopes of distributions; see [9–12].

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